A THEORY OF NATURAL NUMBERS

by

RICHARD B. ANGELL

The Theory of Pure Numbers.

The theory of pure numbers includes

1. Definition of pure numbers, and examination of their properties:
    1.1 positive integers and
    1.2 complex numbers
        1.21 Sums
        1.22 Products
        1.23 Powers
2. Binary relations between pure numbers
    2.1 Sameness Relations (in various respects) e.g.,
        2.11 Identity with respect to having all of a certain set of essential properties.
        2.12 Integral Equality = same in reducibility to the same positive integer
        2.13 Proportional Equality = sameness of ratios
            2.131 Among ratios **If** ((x × y) = (z × w)) **then** (x:z = w:y)
                [E.g., **if** 3×20 = 4×15 **then** (3:4 = 15:20)] or 3/4>15/20
    2.2 Difference Relation (in various respects)
        2.21 Greater, Less among positive integers & compounds of them
            If a & b are PI's, then a>b iff (Ex)(PIx & x + b = a)
        2.23 Greater, Less, among proportional ratios: **If** a×d > b×c **then** a:b > c:d

. 3. Functions and Polyadic relations among 3-tuples and n-tuples.
    3.31 Amounts of Difference Function between natural numbers:
        "D(4,7,3)" for "The difference between 4 and 7 = 3"
        Amounts of difference: E.g., D(4,7) = 3 or |7-4|=|4-7|=3 (no Negative nmbers),
    3.32 Amount of difference in proportion: D((x:y),(z:w)) = (D((x×w),(z×y)):(y×w))
        E.g., D((3:4),(15:21)) = (D((3×21),(4×15)):(4×21)) =3:84
    3.33 Proximation function
        2.132 Relative Proximateness E.g., $(14:10)^2 =_{Prox} 2:1$
            for $14:10 \times 14:10 =_{Prox} 200 \times 100$, $196:100 =_{Prox} 200:100$

4. Sequences An ordered set of numbers, each successive member being determined by some operation on preceding members.

All Sequences a) have a first member, or a first ordered n-tuple of numbers.
    b) each successive member is determined by some fixed operation on one or more preceding members.

.Kinds of sequences: a) Arithmetic. The function is additive. The initial member is a ,
The value of each subsequent member is a function that adds something to some function of preceding members by a general formula.
    Initial member = 1, successor member $a_n = (a_{n-1} + 1)$: 1,2,3,4,...
    Initial member = 2, successor member $a_n = (a_{n-1} + 2)$: 2,4,6,8,...
b) Geometric. Initial member is a, successive members are formed by multiplying one or more specified predecessors, by some formula:
    Initial member = 3, Successor member $a_n = (a_{n+1} \times 3)$: 3, 9, 27, 81,...
    Initial members = 1,2, Successor member $a_n = (a_{n-1} \times a_{n-2})$: 1, 2, 3,5,8,...

Metric fractions, zero, negative integers, irrational roots, imaginary and complex numbers are not "pure numbers"

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For Fla ST Un universe: Erase any time. John suggested I would be interested in this. 12/03.
http://micro.magnet.fsu.edu/primer/java/scienceopticsu/powersof10/index.htm

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QA8.4 .M36 1990 Paolo Mancosu *Philosophy of Mathematics and Mathematical Practice in the Seventeenth Century*

Chapter 1 - INTRODUCTION

Let us distinguish 1) mathematics, 2) *axiomatic theories of* mathematics or (meta mathematical theories), and 3) *philosophies of* mathematics. Mathematics is what it is regardless of theories or philosophical ideas about it's nature. This book presents axiomatic theory *about* a portion of mathematics - the natural numbers. There are alternative theories. Different theories will be more or less compatible with various philosophical ideas about mathematics.

1. *Mathematics* itself is what mathematics teachers teach, what millions of students learn some of and what mathematicians, qua mathematicians are able to do. Millions of school children know that mathematics includes adding, multiplying, subtracting and dividing positive and negative numbers and fractions, finding roots and powers of numbers, and solving problems involving trigonometric functions, logarithms and calculus. Beyond this basic core, mathematics extends into many fields of application, higher mathematics and emerging areas on the borderline . Distinctions are often made between pure and applied mathematics. Although a generalized distinction of this sort is problematic, this distinction is important for a theory of natural numbers.

2. *The Natural Numbers*, or positive integers, consist only of the numbers 1, 2, 3, 4, ..etc., and integers that can be denoted in arabic notation by 10, 11, 12,..., 100, 101,...etc.. Negative numbers and zero are not natural numbers, though the symbol used for "zero", namely '0', occurs prominently with a different meaning, in arabic notation.

3. *An axiomatic theory* about a field of mathematics assumes the pre-existence of the mathematics of that field and its established concepts, principles, equations, algorithms, rules, etc. The theory is intended to integrate these results and show how they can all be derived logically from a set of relatively simple clear concepts, axioms, postulates and rules. Such theories are sometimes called "foundations of mathematics" and sometimes "metamathematics"**;** in either case they are *about* a pre-established field of mathematical concepts and its results.

4. *A theory of natural numbers* is about the field of mathematics that covers only operations, properties and relations of natural numbers. At first sight such a theory would appear to leave out vast areas of mathematics in which the concepts of zero, negative numbers, and many other kinds of numbers (including infinite numbers) that figure prominently in higher mathematics. The truth of this view is just what we want to investigate. Some mathematicians have said,

> "While the Greeks chose the geometrical concepts of point and line as the basis of their mathematics, it has become the modern guiding principle that all mathematical statements should be reducible ultimately to statements about *natural numbers*, 1, 2, 3, ..." [1]

To what extent can this this statement be proved? How far can we go and how do we get there, if our investigations are restricted to natural numbers?

To say a given mathematical statement is *reducible* to a statement about natural numbers is to say that the given mathematical statement (which may appear to be about zero or negative, or complex, or infinite numbers) is provably equivalent or synonymous to another statement that uses only the definitions, axioms, postulates and rules of the theory and talks only about operations on, or properties and relationships of entities that are natural numbers. Obviously this is a tall order.

5. *The adequacy of a theory* for a given field of mathematics is measured by the degree to which the theory is 1) *complete* - in the sense that no significant results in that field are left out of

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[1] Richard Courant and Herbert Robbins, *What is Mathematics*, Oxford University Press, 1941, p 1

account, i.e., all relevant valid statements of established mathematica follow by clear rules of derivation from its definitions and postulates; 2) *consistent* in the sense that it does not yield inconsistent results, and 3) *rigorous*, i.e., its definitions, rules of derivation, etc., have unambiguous, publicly accessible criteria so results will be the same for all who use them. If a theory is also simple and compatible with both ordinary language usage and sophisticated mathematics, this feature may stand in its favor, although it is neither a necessary not a sufficient condition.

6. *Philosophies of Mathematics* are something else again. Philosophy is concerned with general theories of knowledge (epistemology and logic) general theories of value (ethics and aesthetics) and general theories of reality (metaphysics and ontology). Mathematics is just one branch of knowledge, but it is one in which logic plays a central role and ontological questions about numbers and reality are very much alive. One may have strong philosophiical views about mathematics, without having any theory adequate to back them up.

7. In the twentieth century there were three major theories of mathematics. Each necessarily included a theory of natural numbers. a) The Frege-Russell theory sought to derive mathematics from a logic of classes. b) The formalist theory of Hilbert which viewed mathematical statements as results of applying certain rules for manipulating mathematical signs. c) The intuitionist theory of Heyting and Brouwer which asserted that mathematics is based on intuitive ideas and constructions built up from those ideas**.**

8. These theories were guided by different philosophical ideas.The Frege Russell theory appealed to the philosophical idea that mathematics could be derived from abstract concepts and rules of logic which contains the most universal laws of the universe. . The intuitionists appealed to the philosophical idea that mathematics is an activity of the human intellect, and that its foundations must consist of intuitively clear ideas and methods for constructing new ideas from old. The formalists, impressed with the rigor of formal mathematical derivations, sought to avoid philosophical stands about abstract meanings of mathematical symbols or the place of mathematical entities in ultimate reality, by concentrating on the rules for moving from one symbol to another..
..

9. Each of these theories had defects with respect to its adequacy. The intuitionists theories used a more restricted logic than Russell, Frege and Hilbert and were unable to cover certain areas in "classical mathematics" involving infinite sets. Thus they failed the completeness requirement. The Russell-Frege theory conains anomalies with repect to ordinary language, and led to various logical paradoxes which had to be neutralized by *ad hoc* devices in order to avoid inconsistency.But both Hilbert and Frege-Russell claimed to cover all of "classical mathematics" including theories of transfinite numbers. However, Gödel proved that their system of logic could only be complete with respect to mathematics if it was inconsistent, and could only be consistent if incomplete.

10. Despite the absence of a completely adequate theory of mathematics, each of these theories made important contributions. The concepts of an axiomatic system and rigorous proofs was greatly sharpened by both the Frege-Russell theory and Hilbert's investigations of proof theory. The intuitionists kept alive the puzzling question of how human mathematicians can prove statements about non-denumerable infinite sets and numbers that even their author, Cantor, described as "incomprehensible to the human understanding" and knowable only to the Absolute.

11. In the following pages we will begin to develop an axiomatic theory of natural numbers and see how far we can go with the principle that "all mathematical statements should be reducible ultimately to statements about *natural numbers*."

Chapter 2 - POSITIVE INTEGERS AND NUMBERS IN ELEMENTARY ARITHMETIC

Standard alphabets in the western world do not consider parentheses, '(' and ')', as *letters* in the alphabet. Nevertheless, parenthesis signs occur widely in written language, and are universally understood to represent devices which signify groupings of words and clauses. Ordinary language has many other devices (commas, periods, paragraphing, etc.) for grouping. In logic and mathematics the identification of differences in meaning due to differences in groupings is absolutely essential and all structures of groupings are expressible with parentheses. For example, '((2+3)×4)' has a different meaning than '(2+(3×4))' though they differ only in grouping expressed by the different positions of the parentheses. The first expression = 20, the other =24.

The number signs '1', '2', etc., are also not considered as letters in the alphabet, though they are used just as much as words made of alphabet letters to convey information. In the theory of arithmetic below we treat the concept of grouping as the foundation of mathematics. The signs for numbers will be defined in terms of certain purely parenthetical expressions and construed as meaningful words.

2.1 The number 1.

We will use a matching pair of parentheses , '( )', to represent the abstract concept of a single distinct entity. These two marks (without the quotes) are to be viewed as a linguistic expression, useful to convey the meaning we associate with it. This meaning is simply the abstract concept of *a single entity or individual thing, or unit.* The abstract concept of an entity - which is the meaning we assign to '( )' is similarly the abstract concept of some thing, event, idea, etc., whether simple or complex, with boundaries that mark it off from what is *other than it*. It is like the abstract concept of a Gestalt figure distinguished from its background.

Single quote-marks around an expression composed of a pair of matching parentheses signifies, <u>stands for</u> or <u>denotes</u> just those marks (not their meaning) in the order they occur between the single quote-marks in '( )'. These two parentheses enclose a space, set off from other linguistic entities or marks to the right or left the pair of parentheses, and as such each can be viewed as a single whole linguistic entity or word. Thus the sign '( )' is *ideographic*; it displays what it means. Without quote-marks '( )' means "an individual entity" or "unit". The concept of the number *one,* or 1, can be conveyed by the sign, '(( ))'. In this book the sign '(( ))' *means* the number one, the idea of. a group of just one individual entity. Thus we define "one", or the number one, or "1" as:

D1. '1' Syn '(( ))'

and we note that the sign '(( ))' displays what that sign means: it is group with just one entity in it.

2.2 Positive Integer

We use the sign '(( )( ))' to mean the number two; the sign, '(( )( )( ))' to mean the number three, and so on for all positive integers. The idea of a positive integer is the idea of a group of single entities. Each unit in the group is different and distinct from all others in the group. The signs, '(( )( ))' and '(( )( )( ))' not only *mean* the ideas of 1) a group with just two entities in it and 2) a group with just three entities in it; these signs *display* what they mean. The meanings of these two signs each apply to the actual linguistic signs which convey that meaning. The idea of "two" applies to '(( )( ))' because each occurrence of '( )' means 'a unit', so the word "(( )( ))" means "two units" and is itself an instance of what it means.

The clause "**...** and so on for all positive integers" is made precise in the following generative definition of the predicate "â is a positive integer", which we abbreviate as 'PIâ'. However, to distinguish positive integers more clearly from other components of a grouping structure from here on we will use 'o' to abbreviate occurrences of '( )' which occur only in expressions of positive integers, i.e., any pair of left & right parentheses that has no symbols in between them. So '(( )( ))' becomes abbreviated as '(oo)', '(()()())' become '(ooo')etc.. Thus 'positive integer' (abbr. 'PI') is defined as**:**

D2. [â is a PI] Syn [(â is (o) **v** ((ã) is a PI & (ä) is a PI & âis(ã ä)))]

In ordinary English D2 says**:** to say that an entity â is a positive integer *means* ***either*** that â is the same as (o), ***or*** that some entity (ã) is a positive integer and some entity (ä) is a positive integer and â is the same as the result of enclosing the *contents* of (ã) and (ä) in one pair of parentheses**.**

From this definition the following rule for constructing additional positive integers, follows**:**

R1. **If** [(ã)] is a PI & [(ä)] is a PI, **then** & [(ã ä)] is a PI.

Using definition D2, and

Axiom 1 . ((o) is a PI)

we prove that (o), (oo), (ooo), and (oooo) are positive integers as follows**:**.

1)	(o) is a PI	[Axiom 1]
2)	(o) is a PI	[Axiom 1][2]
3)	(o) is a PI & (o) is a PI	[Adj, 1),2)]
4)	(oo) is a PI	[3), Df, PI, Clause(ii)]
5)	(o) is a PI & (oo) is a PI	[Adj, 2),4)]
6)	(ooo) is a PI	[5), Df,PI, Clause(ii)]
7)	(o) is a PI & (ooo) is a PI	[Adj, 2),6)]
8)	(oooo) is a PI	[7), Df,PI, Clause(ii)]
9)	(oo) is a PI & (ooo) is a PI	[Adj, 4),8)]
10)	(ooooo) is a PI	[9),Df,PI, Clause(ii)]

By similar steps any sequence of matched parentheses in which every left parenthesis is immediately followed by a right parenthesis except for the outermost pair which encloses the whole sequence, is an expression that stands for and displays a positive integer**.** Nor need they all be built up using '(o)'. Since (oo) and (ooo) are positive integers [step 4 and step 6], (ooooo) is a positive integer by clause (ii) of definition D2. D2 permits the assertion that no matter how large a given positive integer is, there will be one and only one purely parenthetical expression ("word"), that can

[2] The steps in the following proofs presuppose and omit the following logical steps (1) each step is an implicit truth-assertion; e.g., '(o) is a PI' is an implicit assertion that " '(o)' is a positive integer" is true. (2) that the inference from Pis true and Qis true, to (P&Q) is true is logically valid, and (3) that the inference from 'P is true' to 'It is true that (P or Q)' is valid. [See A-LOGIC, T8-703d, Ti7-783 etc]

have that positive integer as its meaning. Written in the present type font (12 characterer per inch), the “word” for 1,000 (one thousand) would be 2.8 meters long; the word for 1,000,000,000 ()one billion) would be 2,857 kilometers (over 1775 miles). There is no limit to the length of the sign a possible particular positive integer might have. The concept of positive integer is the concept which applies to any one of them -- the concept expressed in the first sentence of this paragraph. To disinguish particular positive intgers from others, we need to find short numerals that will name them.

2.22 Naming Positive Integers, Step 1

Obviously we need shorter expressions to do the work of arithmetic and more signs to distinguish each positive integer from others. We can abbreviate the ideographic terms for positive integers with shorter symbols. These are called numerals.We want a system of numeral-construction that will assign one and only one numeral to each distinct positive integer. Numerals *stand for,* or denote, positive integers. They are used to talk about positive integers. But they are not positive integers themselves and they do not display positive integers the way the parenthetical expressions above do. Nevertheless, using numerals in an arabic system of number notation we can formulate rules such that each distinct numeral, names one and only one distinct positive integer (parentheses grouping that displays it) that we may wish to talk about.

We start with a set of elementary numerals. We choose a small number of positive integers and name each one of them by a distinct single symbol as its numeral. Each abbreviation asserts that the new symbol has the same meaning (is Synononymous by definition to) as the abbreviated symbol**:**

Df ‘1’**:** ‘1’ Syn_{df} ‘[o]’
Df ‘2’**:** ‘2’ Syn_{df} ‘[oo]’
Df ‘3’**:** ‘3’ Syn_{df} ‘[ooo]’
Df ‘4’**:** ‘4’ Syn_{df} ‘[oooo]’
Df ‘5’**:** ‘5’ Syn_{df} ‘[ooooo]’
Df ‘6’**:** ‘6’ Syn_{df} ‘[oooooo]’
Df ‘7’**:** ‘7’ Syn_{df} ‘[ooooooo]’
Df ‘8’**:** ‘8’ Syn_{df} ‘[oooooooo]’
Df ‘9’**:** ‘9’ Syn_{df} ‘[ooooooooo]’

We could go on, inventing additional symbols as numerals for ‘(oooooooooo]’, ‘(ooooooooooo]’ etc., but the definitions of ‘1' to ‘9' are sufficient for our immediate purpose. Each elementary single, numeral is qua sign, called a ***digit*****.** These abbreviations, creating new digits, could be continued as far as we wish. .

Step 2, in naming positive integers consists in selecting some system of notation whereby large positive integers can be named by two or more digits arranged in sequence. This we will do in Chapter 3. But before proceeding to Step 2, we will consider the concept of a natural number and some basic operations, properties and relations of natural numbers, as they relate to the nine positive integers named above.

2.3 Natural Numbers; Operations, Properties and Relations

Positive Integers are natural numbers. But we define 'natural number' more broadly to include also expressions that convey ideas of sums, products and powers of positive integers. Thus expressions like '(3+2)', '(3×2)' and '3^2' will be called natural numbers, as well as more extended expressions (e.g., '(3+2+4)', '(3×2×4)', '$(3^2)^4$' and mixed expressions (e.g., '$((3+6^2)\times(2+4))^{5+6}$'). What all natural numbers have in common is that each one is equal (in a sense defined below) to just one positive integer.

2.31 Definition of "is a natural number"

Natural numbers in this enlarged sense are structures of groupings of individual entities. These groupings can be displayed using only left and right parentheses. The simplest components are positive integers. They are one kind of grouping. But.we define 'Natural number' (abbreviated 'Nn') to include terms with other specific kinds of parenthetical groupings as well:

Df;'Nn': 'â is a Nn' Syn '(â is a PI) (clause (i))
v (ã is a Nn & ä is a Nn & â is [(ã ä)])' (clause (ii))
v (ä is a PI & â is the result of replacing each unit 'o' in ä with a Nn) (clause (iii))

From this definition we can derive the following set of rules for constructing natural numbers:

R2. **If** â is a PI **then** â is a Nn
R3. **If** â is a Nn & ã is a Nn, **then** [(â ã)] is a Nn
R4. **If** ä is a PI & ä is a component of some Nn & â is the result of replacing each unit 'o' in ä with a Nn**), then** â is a Nn

All natural numbers can be displayed or exemplified in grouping structures, constructible by these rules, in which only left and right parentheses occur in accordance with the definition of Nn above.

2.32 Fundamental Operations: Addition, Multiplication, Exponentiation

Operations. The operations of addition, multiplication and exponentiation are precisely and uniquely defined in terms of structures of groupings.

Addition. Within the parenthetiical grouping expression that displays a natural number, an occurrence of '))((' is always interpretable as '+'. For example.

'(1+1)' Syn '((o)(o))', since '1' means '(o)' and ')(' may be interpreted as '+' or addition,
'(2+3)' Syn '((oo)(ooo))', since, '2' means (oo), '3' means (ooo) and ')(' neans '+'.
'(1+2+3)' Syn ((o)(oo)(ooo)), since, **...**etc
'((1+2)+3) Syn (((o)(oo))(ooo)) since, ...etc.

Multiplication. If â is a natural number and ã is a natural number then the result of replacing all occurrences of o in â by occurrences of ã may be interpreted as [(â × ã)] , i.e., as "â times ã" or "the multiplication of ã by â".. For example,

'(2 × 1)' Syn '((o)(o))' since '2' means '(oo)' and '((o)(o))' is the result of replacing each o in 2 by '(o)' i.e., by 1.
'(2 × 3)' Syn '((ooo)(ooo))' since '2' means '(oo)' and '3' means '(ooo)' and '((ooo)(ooo))' is the result of replacing each o in 2 by 3, i.e., by '(ooo)'.

'(3×2)' Syn '((oo)(oo)(oo))' since '3' means '(ooo)' and'2' means '(oo)' and '((oo)(oo)(oo))'
is the result of replacing each o in 3 by 2, i.e., by '(oo)'

Note that ((ooo)(ooo))' i.e., '(2 × 3)', and '((oo)(oo)(oo))' i.e.,'(3×2)' have different grouping structures.

'(3× (2 × 1))' Syn '(((o)(o))((o)(o))((o)(o)))' since, etc..

Exponentiation. If â is a positive integer and ã is the result of replacing all occurrences of o in â by â then ã is $â^{2}$. If â is the initial positive integer and ã is the result of replacing all occurrences of o in the preceding natural number byâ **n-1** times, then ã is $â^{n}$. For example

'3^2' Syn '((ooo)(ooo)(ooo))', since '((ooo)(ooo)(ooo))' is the result of
Step 1. Starting with '(ooo)'
Step 2. Replacing all occurrences of 'o' in Step 1
by '(ooo)' to get '((ooo)(ooo)(ooo))',

'2^3' Syn '(((oo)(oo))((oo)(oo)))' since '(((oo)(oo))((oo)(oo)))' is the result of
Step 1. Starting with '(oo)'
Step 2. Replacing all occurrences of 'o' in Step 1
by '(oo)' to get '((oo)(oo))'
Step 3. Replacing all occurrences of 'o' in '((oo)(oo))'
by '(oo)' to get. '(((oo)(oo))((oo)(oo)))' .

The construction of natural numbers that are sums, products or powers is different from the construction of positive integers.. To construct a new positive integer one adds occurrences of units, i.e, ( ), to other ( )'s within a paair of parentheses, whereas the definition of 'Natural number' always adds second- or higher-level groupings within a pair of parentheses, or replaces '( )' (first-level parentheses) with second or higher-level groupings in a Natural number.

Expressions like 'o' or 'oo' not in paretheses, and '(o(o))' or '((ooo)o)' - i.e., expressions in which o lies in a pair of parentheses with anything other than 'o's, are not well-formed number-expressions according to the rules and definitions of PI and Nn.In short, there is no way to get (o(x))) or ((x)oooo) etc., where '(x)' is a Nn.

|= (1+1) is a Nn.

Proof: 1) (o) is a PI [Df 'PI', Clause(i)]
2) (o) is a Nn [Df 'Nn', Clause(i)]
3) ((o) is a Nn & (o) is a Nn) [2), IDEM, SYN-SUB]
4) ((o)(o)) is a Nn [3), Rule 3, MP]
5) (1+1) is a Nn [4), Df'1', Df '+']

Similarly, 6) From, '(o)' is a Nn & '(oo)' is a Nn it follows that
'((o)(oo))' is a Nn, abbreviatable as (!+2)

and 7) From '(ooo)' is a Nn & '(oo)' is a Nn it follows that
'((ooo)(oo))' is a Nn abbreviatable as '(3+2)'

and 8) From '(oo)' is a Nn, it follows by rules that
'((oo)(oo))' is a Nn and that this is
abbreviatable either as '(2 +2)', '(2×2)' or '2^2 '.

and 9) '((oooo)(((oo)(oo))((oo)(oo))((oo)(oo)))
is a natural number abbreviatable as $(4 + (3 \times 2^2))$ or as (4 + ((2+2)+(2+2)+(2+2))

2.33 Intrinsic Properties of Natural Numbers

Given a single natural number, certain predicates may truthfully apply to it without relating it to any other natural number. Other predicates assert assert a relationship between a given natural number and one or more other natural numbers, or between one kind of natural number and another kind. In the first case, we shall say that the predicate describes an *intrinsic property* of a natural number. In all other cases, we shall say it describes relations between one or more natural numbers and other things. Thus for example, the statements 1)-7) desribe intrinsic properties

1) (()()()()()) is a positive integer.
2) ((()())(()()())) is a pure number
3) ((()())(()()())) is a sum.
4) ((()()())(()()())) is a product.
5) ((()()())(()()())) is a sum
6) ((()()())(()()())(()()())) is an exponential number
7) (((())(()()))((())(()()))) is a product of sums

Monadic predicates that truthfully describe intrinsic properties, without reference to other numbers, mostly describe what we may call grouping structures. The distinctions involved are important, but the statements of most interest predicate relationships between numbers. Even such simple statements as

8) (()()()()()()()) is an odd number

is implicitly a statement about relationships, It means that two is not a factor of (()()()()()()()), i.e., that (()()()()()()()) is not equal to any number with the overall form (()()). Without denying that there are significant monadic predicate that are true or false of natural numbers as we have defined them, we pass on to analysis of relational statements

2.4 Relations Between Numbers

A statement about a relation between numbers will have two or more natural numbers as subject and a binary or polyadic predicate**.** Binary statements have the form [R(n_1,n_2)] where 'n_1,n_2' signifies that we are talking about an ordered pair of two different natural numbers, and 'R' represents the predicate**.** "Two plus two equals four" can be expressed precisely by "Equals (((oo)(oo)),(oooo))"**.** Complex natural numbers, like "(((oo)(o))((oo)(o)))" express terms like "2(2+1)" or "two times one plus one", not statements. Statements about intrinsiice properties of natural numbers, like

2.33 Equality

The basic relation in elementary arithmetic is the relation of equality among natural numbers. What do "(a + b) = c" and "(a × b) = d" mean ? We starts off with addition and multiplication tablec that begin with

+	1	2	3
1	2	3	4
2	3	4	5
3	4	5	6

and

×	1	2	3
1	1	2	3
2	2	4	6
3	3	6	9

Each of these two tables is a short way of to make nine equality statements. These tell us the answers to which sums or products are equal to which, though they do not tell us what '=' means.

We shall say two natural numbers (including sums, products and exponential products) are *equal* if they both reduce to the same positive integer**.** Alternatively, we can say that if two natural numbers both refer to the same number of individual units, they are equal. By "the reduction of ***a*** to positive integer ***b***" we mean the result of eliminating all intermediate groupings (parentheses) between the outermost parentheses and the units. Thus we may define '=' as follows:

.Df '=': 'â = ã' Syn '(â is a Nn & ã is a Nn & ä is a PI
& â reduces to ä & ã reduces to ä)'

And the addition and multiplication tables are explained as follows:

Addition Table	Nn = PI	Multipllication Table	Nn = PI
(1 + 1) = 2 syn	((o)(o)) = (oo)	(1 × 1) = 1 syn	((o)) = (o)
(1 + 2) = 3 syn	((o)(oo)) = (ooo)	(1 × 2) = 2 syn	((oo)) = (oo)
(1 + 3) = 4 syn	((o)(ooo)) = (oooo)	(1 × 3) = 3 syn	((ooo)) = (ooo)
(2 + 1) = 3 syn	((oo)(o)) = (ooo)	(2 × 1) = 2 syn	((o)(o)) = (oo)
(2 + 2) = 4 syn	((oo)(oo)) = (oooo)	(2 × 2) = 4 syn	((oo)(oo)) = (oooo)
(2 + 3) = 5 syn	((oo)(ooo)) = (ooooo)	(2 × 3) = 6 syn	((ooo)(ooo)) = (oooooo)
(3 + 1) = 4 syn	((ooo)(o)) = (oooo)	(3 × 1) = 3 syn	((o)(o)(o)) = (ooo)
(3 + 2) = 5 syn	((ooo)(ooo) = (ooooo)	(3 × 2) = 6 syn	((oo)(oo)(oo)) = (oooooo)
(3 + 3) = 6 syn	((ooo)(ooo)) = (oooooo)	(3 × 3) = 9 syn	((ooo)(ooo)(ooo)) = (ooooooooo)

This way of defining equality works for all Natural numbers, i.e., for all sums, products and exponentiations. For example,

4 +((2^2) (2^2) (2^2))
(4 + (3 × 2^2)) syn ((oooo)(((oo)(oo))((oo)(oo))((oo)(oo))))
reduces to (oooo oo oo oo oo oo oo) syn 16

2.34 Equality Differs from Identity and Synonymy.

The sense of 'Identity' in standard logic is oxymoronic. Wittgenstein said, "To say of *two* things that they are identical is nonsense, and to say of one thing that it is identical with itself is to say nothing at all" 5.5303. TLP. What is involved is similarity, and similarity is a triadic relation, not a binary relation. A is similar to B *wiith respect to* C. We will say that two signs for natural numbers that occur at different times or locations are "identical" if they are similar *with respect to* all internal grouping structures. Thus two sign-occurrences ((oo)(ooo)) and ((oo)(ooo)) are identical but ((ooo)(oo)) is not identical with either. The same criterion holds for other words and signs. ((2+3)=5) and ((2+3)=5) are identical, but no two of ((2+3) = 5),((3+2) = 5) or (5 = (3+2)) are identical. The
differences in identity are clearly established in the expressions using only parentheses. For identity of signs, the components, their order and their grouping must be the same.

. The relation of synonymy obviously does not entail identity. Whether words or phrases are in the same language or in different languages, all significant instances of synonymy are instances of non-idenical signs having the same meaning. '1' is not identical with '(( ))', and '((oo)(oo)) = (oooo)' is not identical with '(2+2)= 4', though both pairs are synoymous. In logic (Pv(QvR)) is synoymous with ((PvQ)vR) but the two formulas are not identical. It is a desideratum of language that identical expressions used in a normal situation should have one and the same meaning. This is a requirement in any logical argument.. So for logical and mathematical systems we may say that identical expressions within those systems must be synonymous. The converse does not hold.

Among natural numbers, equality (as defined above) does not entail identity either of the terms used or of the referents of those terms. ((oo)(oo)(oo)) is equal to ((ooo)(ooo)) because both reduce to the same positive integer, (oooooo). But they are not identical because their internal groupings are different. And they are not synonymous. Two boxes of three chocolates is not synonymous with three boxes of two chocolates. Thus although (2×3) = (3×2) it is not the case that (2×3) is identical to (3×2), or that '(2×3)' syn '(3×2)'.

Interestingly, (2 + 2), (2 × 2), and 2^2 are all synonymous since all are synonymous with ((oo)(oo)), and all are equal, though no two of the four expressions are identical. These synonymies are unique to the number 2, and are not characteristic of +, × and the second power generally. None of '(3 +2)', (3 × 2), and 3^2 are either synonymous or equal, though (3 × 2) is synonymous with and equal to (2+2+2) and 3^2 is synonymous with and equal to (3+3 +3) according to our defintions.

2.35 Other Properties, Relations and Operations

Many other properties and relations of natural numbers can be defined based on definitions above.

The relation, 'â is the successor of ã' abbreviatiated as 'S(â,ã)' can be defined as,

[S(â,ã)] syn$_{df}$[(ã) is a PI & â is (ã())]

The successor relation is the basis of the concept of enumeration or counting.

The relations "â is greater than ã"(abbreviated as 'â > ã') and "ã is less than â" (abbreviated as 'ã < â ') can be defined in terms of addition:

[â > ã] syn$_{df}$ [ã is Nn & (Ex)(x is Nn & â = (ã+x))]
[â < ã] syn$_{df}$ [ã > â]

'x>y' for '(Ez)(x = (z + y))'
'x<y' for '(Ez)(y = (z + x))'

Diff<â,ã,ä> for "The arithmetic difference between â and ã is ä"
"PIâ & PIã & PIä & (â + ä)= ã"

D2 '|x-y| = z' for 'x = (z + y)'
'|y-x| = z' for 'y = (z + x)'

D3 |x-y| = z & ~(Ew)(Ev)(|x-w| = v & v<z'
y is the closest number to x that is smaller than x

D4 |y-x| = z & ~(Ew)(Ev)(|y-w| = v & v<z'
y is the closest number to x that is larger than x

The concept of "â is a factor of ã" (abbreviated 'Fâã') or "ã is divisible without remainder by â" is definable in terms of multiplication:

[Fâ ã] syn_{df} [(Ex)(x is PI & ã = â×x)]

and from this we can define the properties of being divisible by 2 (being Even) or by any other number, and of "â is a Prime number" (abbreviate 'Prâ')

[Prâ] syn_{df} [~(Ex)(x is PI & ~ x=â & Fâ x)]

Finally, although the natural numbers do not include negative numbers or any numbers smaller than 1, the concepts of subtraction and division can be introduced as complements of addition and multiplication. Thus, subtraction relative to a sum, is defined as

|= [If ((â + ã) = ä) then (â = (ä&ã))]

This rule allows transfers of a number from one side of an equation to the other provided the + sign is changed to & . Similarly, division is introduced relative to a product.

|= [If ((â × ã) = ä) then (ã = (ä ÷ â))]

As for the concepts of there being no number x, such that x = â□â when â is a Nn, or such that x = ã□â when â > ã; this is simply stated (instead of a number "zero" or a negative number) by the postulate;

|= [If â is a Nn & ã is a Nn & (â > ã v â = ã)
then ~(Ex)(x is Nn & ((x = (ã ÷ â)) v (x = ã& â)))]

2.5. Ratios' Relations and Properties (Rational Numbers)

If we wish to assert that two numbers stand in a certain relation, we present them as an ordered pair, (n_1,n_2) this ordered pair becomes the subjects of a predicate. E.g., "Is greater than", "is Less than", "is integrally equal to", "is a factor of...". In English we usually put these predicates between the two subject terms as in "n_1 is greater than n_2 ", "n_1 is Less than n_2 ", "n_1 is integrally equal to n_2", etc., and this is reflected in the conventional mathematical language: $(n_1 > n_2)$, $(n_1 < n_2)$, $(n_1 = n_2)$.. Because mathematics has many predicates that apply to ordered triples, ordered, n-tuples, and we want to regularize the form of predicates with the predicates preceding the subjects., we might use the notation $>(n_1,n_2)$, $<(n_1,n_2)$, $=(n_1,n_2)$., but we shall avois this untill it becomes necessary.

In many cases we want to talk about the relations between two or more ordered pairs of numbers, rather than about the relation between members of one ordered pair of natural numbers.

The statements "2 is less than 4" and the "3 is less than 6", each talk about a relation between two numbers in an ordered pair. But we may wish to point out that the ordered pairs (2,4) and (3,6) have a cetain relationship, e.g., their two members stand in the same proportional relation as (1,2). When we take ordered pairs of natural numbers as subject terms instatements about relations between them, we call them ratios, and ratios of of numbers are expressed as $(n_1:n_2)$ Thus we may say that "The ratio of (1:2) is similar to the ratio of (3:6) in the proportions their members bear to one another." One kind of relation between ratios, let us call it "proportional equality" or '$_P$=', can be defined rigorously using integral equality at the base, .

2.6 General Comment on Natural Numbers

Concepts of natural numbers are independent of properties and relations of things they may be numbers are "of". The number *of* (physical) things in my pocket is 15, namely a ring of keys, a comb and 13 coins. The number *of* coins in my pocket is 13, of which 2 are worth 25 cents, 2 are worth 10 cents, 6 are worth 5 cents and 3 are worth 1 cent. The total value *of* the coins in my pocket is 103 cents. Last night I woke up at 3:00 a.m. and focused my attention on 3 things: 1) trying to calculate the product of 371 times 752 in my head, 2) trying to list reasons why the 2002 war by U.S. against Iraq was wrong, and 3) the diversity of kinds of things that a positive integer could be the number *of.* (i.e., what I am writing about in this paragraph)..

Thus the different kinds of individual entities that can be grouped into one groups to which a number may be applied are of the utmost diversity. The individual entities may be different kinds of physical object (coins, combs, key-rings) different kinds of ideas (ideas of numbers, of reasons for a war), values of coins, of past events, ideas of events or any misture of any of these than may occur, an unlimited other diverse kinds of entities.

Numbers of things are not neceesarily like numbers of inches or numbers of pennies, or centimeters, or numbers of degrees of temperature or numbers of ounces of weight, where all of the units are equal, and adding them up give other units - feet, dollars. pounds, kilometers. The concept of a metric system is an add-on to numbers. Pure natural numbers apply to entities, and three entities consisting of a penny, an elephant and the Hudson bay - or the 23 diverse items my wife brought home from shopping - don't add up to one thing the way twleve inches add up to one foot, and 3 inches add y\up to a quarter of one foot. The concept that every point on a straight line can be assigned a number - the contiuum hypothesis - may be a desideratum, but it does not come from the concept of number or equality of numbers, but rather from the fascinating equalities of distances and areas and ratios in geometry. Geometrical concepts should not be conflated with number theory.

2.4 Digresssion #1 - Other Theories of Natural Numbers

2.41 Peano

All of Peano's first eight axioms. (postulates?) for elementary arithmetic, will be derivable from our definitions of numbers (the ideographic denotata of numerals as defined in parenthetical notation above), of '1', of '+', and of '=', together with principles drawn from the logic of identity**.** His axioms 2,3,4, and 5 simply apply to numbers the following principles of identity**:**

2. (x) (xIx)
3. (x) (y) (xIy o yIx)
4. (x) (y) (z) ((xIy.yIz) ; xIz)
5. (x) (y) (z) ((xIy . y is an N) ; x is an N)

His first postulate,

1. 1 is an N

translated into our system, says simply that (o) is an number, and since '(o)' is a numeral, and the class of numerals, on our account, stand in one-to-one correspondence with the class of (elementary) numbers, our account satisfies

P1. The postulate P6, says that if any entity is a number then its successor is a number**:**

P6. (x) (x is an N ; (x+1) is an N)

Proof: 1) x is an N	Assumption
2) (oo) is an N	Def. 1
3) (x(o)) is an N	1, 2, n of 2 clause (ii)
4) (x+1) is an N	Df '1' Df '+'
5) If x is an N then (x+1) is an N	(2-4) C.P.

and this certainly will follow also from our definition of numerals and T2. Postulate P7 says that successors of equal numbers are equal.

P7. (x) (y) ((x is an N and y is an N); (x=y o ((x+1)=(y+1)))

and this also is subject to a quick and easy proof. For if x and y are equal, then they reduce to the same positive integer (by the definition of 'R' and '='; the successor of each of these will be denoted by a parenthetical expression which adds only one more zero-level expression, so that when the intermediate parentheses are eliminated both will again reduce to the same positive integer, i.e., the integer which is equal to the successor of the integer they both reduced to in the antecedent. Actually, this postulate will need a principle of induction for its derivation. Postulate P8 says simply that 1 is not equal to the successor of any (elementary) number:

P8. (x) (x is an N ; ((x+1) (1))

And this, also, is perfectly obvious in terms of our definitions of 'N', '1' and '='; (o) is not equal to any [(n(o))] where n is a number. The final postulate, is in effect the principle of mathematical induction:

P8. (x) (x is G ; x is an N) . 1 is G . (x)((x is an N . x is G); (x+1) is G));
6 (x) (x is an N ; x is G).

Although this principle will be a meta-theorem of our system, it is not adequate as a postulate of the system due to the fact that we have not defined numbers in our system solely in terms of the successor function. Definitions D1 and D2 do not yield a linear (or strict simple) ordering as the successor relation does in Peano's definition of N based on P6. For clause (ii) of D2 allows an indefinite number of results of performing the operation of "replacing each occurrence of '0' in a given numeral by some or other numeral". Thus this mode of generating the numbers gives at best a partial ordering. Nevertheless, it is possible to find an alternative to P9 which is stronger, and from which P9 may be deduced.[3] Thus we may conclude that given our definition of the relation, as an

[3] 1 Cf. For example, Kleene, Stephen Cole Introduction to Metamathematics, 1952 s 50, in which he indicates how to formulate an appropriate principle of induction for the system of Hans Hermes' "Semiotik Eine Theorie der Zeichengestalten als Grundlage für Untersuchungen von formalisierten Sprachen", Forschungen zur Logik und zur Grundlegung der exakten Wissenschaften, n.s., No. 5, Leipzig, 1938. Hermes definition of 'entity', like our definition of number, involves parenthetic enclosure of any finite series of entities previously established, and thus yields only partial ordering.

operation on compound groupings and the consequent definition using logical identity, I, and predicate logic of '=', our theory of elementary arithmetic will conform to the requirements of Peano arithmetic, and thus constitute a viable theory of the arithmetic of positive integers.

We could go further and define other functions - the subtraction function, the division function, - and relations like greater than, less than, etc., in familiar ways with the concepts at hand. More ambitiously, we could present a formal axiomatized system, and set about proving this system complete with respect to Peano Arithmetic. However, these tasks are not relevant to our present purpose. Our purpose thus far has been merely 1) to define a domain of objects, called elementary numbers (which may be called a sub-class of the class of organized groupings), 2) to give an inductive definition of a specific set of linguistic expressions to be built up from parentheses '(' and ')' and called 'numerals', 3) to propose that all and only those organized groupings which correspond ideographically one-to-one to these numerals constitute the exact domain of objects that elementary arithmetic is about, and 4) to propose that the relations: and function based on the elimination of all intermediate groups, Rxy, (as distinct from the successor function) together with logical identity and predicate logic, may be adequate for all the relations, beginning with arithmetic equality, '=' needed in elementary arithmetic. This we will examine in the next section.

Our final step is to show, if that this much as been granted, then it is possible to define in a very clear way the difference between mathematical relation and certain kinds of relations between numbers or sets of numbers which are contingent and non-mathematical.

2.42 Frege-Russell

Frege and Russell wanted to derive all of mathematics, including of Cantor's theory of infinites, from logic - from the logic of classes in particular. Central to their project was Cantor's concept of equipollence. Two sets are equipollent if and only if there is some function or way to establish a 1:1 correspondence between their members - i.e., for every member of the first set there is one and only one member of the second, and for every member of the second set just the one corresponding member of the first. If two sets are finite then they can be equipollent if and only if they have the same number of members. For example the sets of numbers {1,2,3} and {2,7,8} are equipollent, since they can be put in 1:1 correspondence in several ways. E.g., the pairing {<1,2>,<2.7> ,3,8>} or the pairing {<1,7>,< 2,8>,<3,7>} are .equipollent., but {1,2,3} and {1} are not. Infinite set, however, can be equipollent when one is only a subset of the other. The set of even numbers is equipollent with the set of all natural numbers because for each even number there is one and only one natural number that is half of it and for each natural number there is one and only one number that is twice that number. But of course there are many numbers (odd numbers) in the set of natural numbers that are not members of the set of even numbers - in a certain obvious sense, the set of natural numbers is greater than the set of even numbers, so the two sets are equipollent but not identical.

The Frege-Russell definition of a cardinal number (vs an ordinal or counting number) is simple to state. The cardinal number A of the set A, is the class of all sets equipollent to A. Using ' . ' for the relation "...is equipollent to.." this is symbolized as

The number A = {B: B . A}

Frege and Russell then define the numbers 0, 1 and 2, as follows (we subscript these numbers with *f* to indicate these are Frege-Russell definition of these numbers):

$$0_f =_{df} \{0\}$$

that is, in the Frege Russell view is the set of all sets having no members. The number 1 is defined as,

$$1_f =_{\mathrm{df}} \{A\mathbf{:}\ (Ex)(x0A\ \&\ (y)(y0A\ q\ x= y))\}$$

that is, the number 1 is the set of all unit sets (all sets having only one member). In the Frege Russell system this is equivalent to the following definition using equipollence:

$$1_f =_{\mathrm{df}} \{A\mathbf{:}\ A\ .\ \{0\}\}$$

I.e., 1_f is the class of all classes that are in 1**:**1 correspondence with {0}

The definition of 2_f (in contrast to $2_a\,\mathrm{syn}_{df}$ (()())) is:

$$2_f =_{\mathrm{df}} \{A\mathbf{:}\ A\ .\ \{0\{0\}\}\}$$

or, equivalently,

$$2_f =_{\mathrm{df}} \{A\mathbf{:}\ (Ex)(Ey)(x0A\ \&\ y0A\ \&\ x...y\ \&\ (z)(z0A\ q\ (z=x\ v\ z=y))\}$$

and 3_f (in contrast to 3_a $\mathrm{syn}_{\mathrm{df}}$ (()()())) is defined as

$$3_f =_{\mathrm{df}} \{A\mathbf{:}\ A\ .\ \{0\{0\{0\}\}\}\}$$

$$\text{or } 3_f =_{\mathrm{df}} \{A\mathbf{:}\ (Ex)(Ey)(Ew)(x0A\ \&\ y0A\ \&\ w0A\ \&\ x...y\ \&\ y...w\ \&\ w...x\ \&\ (z)(z0A\ q\ (z=x\ v\ z=y\ v\ z=w))\}$$

Thus the class {George Washington, Abraham Lincoln, Franklin Roosevelt} 0 3_f; i.e., the class of those three individuals is a member of 3, which is the class of all classes that have just three members.

The equipollence definition of number allows Frege-Russell to assign a number to the infinite class of all natural numbers, namely the cardinal number T_1**.** Other infinite classes have different cardinal numbers**.** This is cumbersome way to define number set in a paradox-laden theory of classses. But with a little ad hoc patching up, it is remarkable for its capacity to produce proofs about its sets which corresponded to accepted mathematical results; even though Gödel showed it could never be both complete and consistent.

2.43 Hilbert

In 1922 Hilbert published "The New Grounding of Mathematics; First Report". Like Frege and Russell he wanted to have an axiomatic foundation for mathematics that would cover all of "higher" mathematics including Cantor's numbers for infinite classes. But he rejected the definition of number through the logic of set theory on the grounds that "the concept of a set has given rise to paradoxes" (p 199). "Frege tried to ground number theory on pure logic," he wrote, "Dedekind tried to ground it on set theory as a chapter of pure logic**;** both failed to reach their goal."(p 201) However, he pointed out that "the paradoxes of set theory can not be regarded as proving that the concept of a set of integers leads to contradictions".(p 199) He went on,

"The solid philosophical attitude that I think is required for the grounding of pure mathematics...is this: *In the beginning was the sign.*

"With this philosophical attitude we turn first to the theory of elementary arithmetic, and ask ourselves whether and to what extent, on this purely intuitive basis of of conrete signs, the science of number theory coujld come into existence. We therefore begin with the following following explanation of the numbers.

"The sign 1 is a number.

"A sign that begins with 1 and ends with 1, and such thatin between + always follows 1 and 1 always follows +. Is like a number; for example the signs

1+1
1+1+1

These number signs [*Zahlzeichen*], which we assume are numbers and which completely make up the numbers, are themselves the object of our consideration, but otherwise they have no *meaning* [*Bedeutung*] of any sort. In addition to these signs, we make use of yet other signs that *mean* someth and servee for communication, for instance the sign 2 as an abbreviation of the number-sign 1+1 of the sign 3 as an abbreviation for the number sign 1+1+1'; moreover, we use the signs =, >, whhich serve for the communication of assertions. Thus 2+3 = 3+2 is not to be a formujla, but merely to serve to communicate the fact that 2+3 and 3+2, with respect to the abbreviations we are using, are the samne number-sign 1+1+1+1+1. ....

"For purposes of communication we shall also use letters **a**, **b**, **c** for number-signs. Then **b** > **a** is also not a formula, but only the communication that the number sign **b** extends beyond the number-sign **a**. ....

"When we develop number theory in this way, there are no axioms, and no contradictions of any sort are possible. We simply jhave concrete signs signs as objects, we operate with them, and we make contentual [*inhaltliche*] statements about them . And in particular, regarding the proof ...that **a** + **b** = **b** + **a**,,I should like that this proof is merely a procedure that rests on the construction and and deconstruction of number-signs and this is essentially different from the principoe that polays such a prominent role in higher arithmetica, namely the principle of complete induction or of inference from n to n+1. This principle is is rather, as we shall see, a formal principle that carries us farther and and that belongs to a higher level; it needs proof and the proof can be given.

"We can of course make considerable further progress in number theory using th intuitive and contentual manner of treatment which we depicted and applied. But we can not conceive the whole of matheamtics in such a way. Already whenwe cross over into higher arithmetic and algebra - for example if we wish to make assertions about infinitely many numbers of functions - the contentual procedure breaks down. For we can not write down number-signs or introduce abbreviations for infinitely many numbers....

"But we can achieve and analogous point of view if we move up to a higher level of contemplation, from which axioms, formulae, and proofs of the mathematical theory are themselves the object of contentual investigation. But for this purpose the usual contentual ideas of the the mathematical theory must be replaced by formulas and rules, and imitated by formalisms. In other words, we need to have a strict formalization of the entire mathematical theory..." pp202-204

Hilbert goes on to lay out the specific basic signs that are used in mathematics, and to try to develop the formal rules and and formulas that are implicitly used in proofs that move from one mathematical statement to another. In effect it is treating mathematics as rules for manipulating symbosl without reference to what ths symbols mean, but with the requirement that the whole axiomatic system by both consistent and complete. Hilbert's logic was essentially the first order logic of the Frege-Russell-Whitehead, but did not try to define numbers as sets - there were set of numbers, but numbers weren't defined as a certain class of classes of classes.

Hilbert's program, like the Frege Russell program, failed due to Godel's proofs. (But these proofs were relative to systems of logic like PM's .

In my theory ‘1’ has a meaning. It denotes a concept. ‘2’ and each positive iteger denotes a different concept. The relation of. equality, ‘=’. in the theory of natural numbers holds of two numbers only if a certain specific operation one those numbers yields a specific testable result.

2.5 Digression #2 - Applications and Add-ons

The properties and relations of natural numbers *qua* numbers are independent of the properties and relations of things to which we apply those numbers. Applications to particular kinds of things become facilitated by adding on distinctions and concepts that belong the field of application rather than to natural numbers as such. Many such add-ons are based on applying numbers to concepts of Euclidean space and of time which are quite independent of the concept of natural number. We will revert the this theme at each stage, but initially we look upon negative numbers, fractions and the number zero, as examples of this thesis.

2.51 Negative Numbers

Natural numbers are positive, and one can define the *difference* between two natural numbers in terms of addition. If (Î + Ï) = Ð, then the difference between Î and Ð is Ï and the difference between Ï and Ð is Î . This can be expressed as,

If Î is a Nn & Ï is a Nn & Ðis a Nn & (Î + Ï) = Ð **then** Ï = |Ð□ Î | & Î = |Ð□ Ï |.

This may be viewed as subtraction, applied to natural numbers it is like saying

2+3 = 5 (which means) ((oo)(ooo))= (ooooo)

therefore the *difference* between 5 and 2 is 3 and the difference between 5 and 3 is 2 ; “take away (subtract) 3 from 5 and you have 2 left”

|(ooooo)-(oo) | = (ooo)
and |(ooooo)-(ooo)| = (oo)

But given the condition stated in the antecedent, it makes no sense to speak of any positive number that is left, if you try to take a larger number away from a smaller number. Take away 5 from 2 and what do your have? Well, you ***can’t take*** 5 away from 2, because 2 doesn’t contain five units to be taken away. To “take away five units” if you have only two to begin with is an oxymoron, unless you believe in magic.

Although the *difference* of two natural numbers is conditioned upon subtracting a smaller number from a larger number. Mathematicians wanted to generalize subtraction. Instead of the restricted rule (x)(y)(z) (**If** (x+y) = z **then** (x = (z-y) & y= (z-x))), they wanted a general rule, so that any number on one side could be switched to the other side by changing + to□ or vice versa. That is they wanted,

. (x)(y) (If (x = y) then (x□y) = 0 & □y =□x & 0 = (y□ x)

Combined with (x)(y)(z) ((x = y) iff (z+x) = (z+y) which holds of all natural numbers, this allows many new equations by treating negative numbers and 0.as numbers.

The concept of negative numbers is useful in many applications. In commerce we are interested in comparing what we spend to what we owe. If my bank account shows a negative number of dollars, it means I have spent more than I have. In temperature readings, we are interested in whether the temperature will freeze or boil water. We say it is 0 degree centigrade if it is freezeing, and -10 degrees if it is ten degrees below freezing. If we are counting years on a calendar, we shoose some year (e.g., the year of Christ's birth) as the year 0, and then use negative numbers -300 for years before Christ.

In the Cartesian coordinate system which has been so central in what mathematicians call "analysis" (as in analytic geometry), we list two sets of negative numbers (on the x axis and on the y axis), and assign 0.to a point at which positive number end and negative numbers begin. This system allows us to associate 0 in the coordinate system with any particular thing, and then think of negative numbers as representing points going in oppositie directions from the positive numbers on a two dimensional plane. This device is enormously helpful in representing geometrical; shapes, areas, motions, changes in speeds, etc.. of two variables. But it does not come out the theory of pure natural numbers. It is a device which is helpful in applying numbers to geometrical concepts that are quite independent of the concept of natural numbers

Mathematicians have conflated the pure theory of numbers with operations, properties and relations which hold in the field of geometry. Because.in Euclidean geometry the area of a square figure can always be thought of having four equal sides, and these sides can be divided into any n equal legnths, and further the area of that square can then be divided into n×n equal square units which in total equals the area of the initial square, the relation of numbers when applied to geometry are conflated with the relations of natural numbers by themselves, and mathematics is
treated as the merger of geometry and natural numbers. But this is a mistake. Euclidean geometry is enormously useful in human affairs, as is the application of mathematics to its findings. But this does not warrant expanding the concept of number to include zero and negative quantities, the roots of prime numbers, and pi as additional kinds of numbers.

2.52 Division and Fractions

Just as one can not subtract a larger number from a smaller one, one can not *divide* a natural number by a larger number than itself. But if we begin with a product equal to some positive integer, it will follow that that integers is devisible into a specifc number of equal parts.

If Î is a Nn & Ï is a Nn & Ðis a Nn & (Î × Ï) = Ð **then** Ð/ Ï = Î & Ð/ Î = Ï .

where 'Ð/ Ï ' means 'Ð is divisible by Ï '. This does not mean that every number can be divided by any number. We shall deal later with ratios between natural numbers, and much can be said about that. But the idea that any line segment can in theory be divided in to n equal parts comes out of geometry, not from the concept of natural numbers. Even when we get to ratios, the concept of equality of a group of fractional parts, is an add-on from processes of trying to apply numbers to procedures of geometrical construction. It is not an idea derivable from the idea of equality among natural numbers.

Chapter 3 - SYSTEMS OF ARABIC NOTATION

We gave names '1', '2',..., '9' to the positive integers '(())', '(()())'... '(()()()()()()()()())'. But the definition of positive integer allows additional positive integers without limit. We can't continue to use a new single sign for the name of each additional integer**.** This would go on indefinitely. The system of arabic notation uses strings of digits to ascribe a precise name to any conceivable positive integer no matter how large. This method of naming suggests that in theory a a distinct name to can be given to each member in an infinite set of distinct positive integers. In practice, however, that is impossible.

Using just the positive integers from 1 to 9, we were able to give examples of the major operations, addition, multiplication and exponentiation, as well as the major relations (identity, equality, greater than, and successor of) and the major properties (odd, even, prime) of this small set of elementary numbers. With arabic notation the same disitinct operations, relations, properties, rules and algorithms can be shown to be applicable to any natural numbers we may choose - i.e., to natural numbers uniiversally, no matter how large or complex.

3.1 The Decimal System

Our decimal system of numbers with base 10 came from India. It was developed by Arabic scholars, and was introduced in Europe by Leonardo of Pisa, also called Fibonacci, in 1202, although its general use began only around 1500. The first sentence of the first chapter in Fibonacci's book, *Liber Abaci,* is,

> "These are the nine figures of the Indians
> 9 8 7 6 5 4 3 2 1
> With these nine figures, and with the sign 0..., any number can be written as will now be demonstrated."

Instead of a new single digit for the number ten, the first digit '1' was coupled with the sign '0'. The sign '0' is a digit, though it is not the name of any positive integer or natural number standing by itself. Thus the combination '10' - with '1' on the left and '0' on the right - is offered as the name of the positive integer {(oooooooooo). Thereafter no new digits are introduced. Instead, subsequent positive integers are named by ordered sequences of two or more of the ten digits.

The decimal system involves, first, the ordered sequence, 1,2,3,4,5,6,7,8,9, standing respectively, for the positive integers (o), (oo), (ooo), (oooo), (ooooo), (oooooo), (ooooooo), (oooooooo), (ooooooooo). The next positive integer, (oooooooooo), is called the base of the decimal system and is represented by '10'. Secondly, after 10 the 2-digit numbers 11, 12, 13, 14, 15, 16, 17, 18, 19 represent 10+1, 10+2, etc,, i.e, the natural numbers ((oooooooooo) (o)), ((oooooooooo) (oo)), etc., up to 10+9, or 19 for ((oooooooooo) (ooooooooo)). Each of these is equal to (reducible to) just one positive integer, namely, respectively.(ooooooooooo); . (oooooooooooo), (ooooooooooooo), etc., up to (ooooooooooooooooooo). And the arabic numerals, '11', '12', ...,'19' are taken as the names of these positive integers. These positive integers are ordered by increasing size, each number in the sequence being the successor (having one more unit than) the number before it. After 19, we get ((oooooooooo) (oooooooooo)) i.e., 10 + 10, which is two tens, called ‘20’. Thus ‘20’ becomes the name of the positive integer (oooooooooooooooooooo) that is equal that is equal to 2 × 10, ((oooooooooo)(oooooooooo)). 20 is followed by 21, which stands for (((oooooooooo) (oooooooooo))(o)) or 20 +1, then 22, 23,...,29, after which comes 3 × 10 which is means ((oooooooooo) (oooooooooo)(oooooooooo)) ‘30' denotes the positive integer (oooooooooooooooooooooooooooooo) to equal to 3 × 10. And so on. When we get to 99, we have nine 10's plus 9. The successor of 99 is gotten by replacing the right-most positive integer by its successor 10. This gives us ten tens, or (10 × 10) or 10^2, which is equal to (reduces to) the positive integer we call 100.. In this procedure, each successive numeral stands for the positive integer with just one more unit than the one preceding it. One can go on as far as one wishes, with 1000 the name of the positive integer equal to 10×10×10, etc .. .

Children in their first year of school can learn the system of counting to 100, or 1000 ot 10,000 using the decimal system. They memorize the count to 10, and then learn to use that count in moving from ten to twenty, from twenty to thirty, and so on up to 100, to 1000 etc..Learning to count involves a process of memorization plus an application of simple rules that are quite independent of knowing what the numbers mean. That a child can count to 200 in the decimal number system does not imply that he or she knows how to construct any of the positive integers.

Children in early grades are taught algorithms for adding and multiplying numbers as large as you please. These are based on memorizing the addition and multiplication tables for numbers 1 to 10 in the decimal system, together with rules of “carrying over” for sums or products greater than 10. Seldom, if ever, do they produce the actual positive integer which is denoted by the sum or product, but it can be proven that if the algorithm is followed, then the numeral that results at the end of the process, is the name of precisely the positive integer that would be equal to the natural number that is the sum or product of the two initial large numbers.

Given any numeral in the decimal system to find the positive integer it stands for one can proceed as follows: Given the sequence of n digits (without a decimal point) which is the numeral

Step 1) multiplying the left-most digit by the n^{th} power of 10
Step 2) adding the result of multiplying the 2nd left-most digit by the $(n-1)^{th}$ power of 10;
or if the 2nd left-most digit is ‘0’ do nothing and move on,
. . . .
Step (n-1) adding the result of multiplying the 2nd left-most digit by the 1^{st} power of 10;
or if the $(n-1)^{th}$ left-most digit is ‘0’ do nothing and move on,
Step (n) adding the positive integer named by the last (n^{th}) digit
or if the n^{th} left-most digit is ‘0’ do nothing.
Step (n+1) Removing all intermediate parentheseses from the result of steps 1) to n).
The result will be the positive integer named by the given sequence of digits.

3.2 Arabic Notation Generalized for Any Integral Base

The decimal system is one system among many which use the same general principles for naming positive integers. Any positive integer greater than 1 can be the base of an arabic notational system. Presumably (oooooooooo) was chosen as the base because this is the number of fingers humans have, and human counting is aided by pointing to things with our fingers.The general nature of arabic notation is more easily grasped in arabic number systems with smaller bases such as 2, 3, or 6.

The simplest system of arabic notation uses the base 2. The numeral 2 never occurs in numerals of this system because '10' is used in place of 2. In this system the positive integers are enumerated as shown in the table below, as compared with the decimal system

All arabic numeral systems use the digits '1' and '0' and (in English) all have the words "one" for '1', "ten" for '10' and "eleven" for '11', "one hundred", "one hundred and one", "one hundred and ten", "one hundred and eleven", for '100', '101', and '111', "one thousand" for '1000', "one million" for '1,000,000' and so on.

But obviously, these words will have different meanings depending on which base is used. If (oo) is the base, '10' i.e., 'ten', means what we call 2 in the decimal system and '1,000,000' i.e., "one million" means the positive integer associated with 2^6 (called 64 in the decimal system). We need different signs to convey these differences in meaning. To do this we will subscript a simple name for just the positive integer which is the base after each numeral in the given arabic notation. Thus $1{,}000{,}000_2 = 64_X$, where X is the Roman numeral for the number after nine - the positive integer that we who use the decimal system, call "ten". But we must distinquish 'X' or 'decem' from '10' and 'ten'. which have different meanings depending on the base used, In Latin 'X' and 'decem' have just one meaning**:** namely, the positive integer, (oooooooooo). In ordinary language 'ten' and 'X' are thought of as synonymous. This is because we customarily use of the decimal system. But they are not synonymous here. We want 'X' to mean only the positive integer (oooooooooo), while 'ten' and '10' means the base of whatever arabic numeral system is being used. Let 'Z' stand for 'twelve', i.e., (oooooooooooo). In each row of the table below, the positive integer in the left-most column displays the meaning that is common to all of the different names given to in different arabic numeral systems shown in the five columns on the right.

Positive Integer	Name in 2-system and in English		Names in 3-system	Names in 6-system	Names in X-system	Names in Z - system
(o)	1_2	One	1_3	1_6	**1_x**	1_z
(oo)	10_2	Ten	2_3	2_6	**2_x**	2_z
(ooo)	11_2	Eleven	10_3 ten	3_6	**3_x**	3_z
(oooo)	100_2	One hundred	11_3	4_6	**4_x**	4_z
(ooooo)	101_2	One hundred and one	12_3	5_6	**5_x**	5_z
(oooooo)	110_2	One hundred and ten	20_3	10_6 ten	**6_x**	6_z
(ooooooo)	111_2	One hundred and eleven	21_3	11_6	**7_x**	7_z
(oooooooo)	1000_2	One Thousand	22_3	12_6	**8_x**	8_z
(ooooooooo)	1001_2	One thousand and one	100_3	13_6	**9_x**	9_z
(oooooooooo)	1010_2	One thousand and ten	101_3	14_6	**10_xten**	X_z
(ooooooooooo)	1011_2	One thousand and eleven	102_3	15_6	**11_x**	Y_z
(oooooooooooo)	1100_2	One thousand one hundred	110_3	20_6	**12_x**	10_z ten

(ooooooooooooo)	1101_2 One thousand one hundred and one	111_3	21_6	$\mathbf{13_x}$	11_z
(oooooooooooooo)	1110_2 One thousand one hundred and ten	112_3	22_6	$\mathbf{14_x}$	12_z
(ooooooooooooooo)	1111_2 One thousand one hundred and eleven	120_3	23_6	$\mathbf{15_x}$	13_z
(oooooooooooooooo)	10000_2 Ten thousand	121_3	24_6	$\mathbf{16_x}$	14_z
(ooooooooooooooooo)	10001_2 Ten thousand and one	122_3	25_6	$\mathbf{17_x}$	15_z
(oooooooooooooooooo)	10010_2 Ten thousaand and ten	200_3	30_6	$\mathbf{18_x}$	16_z
(ooooooooooooooooooo)	10011_2 Ten thousand and eleven	201_3	31_6	$\mathbf{19_x}$	17_z
(oooooooooooooooooooo)	10100_2 Ten Thousand, one hundred	202_3	32_6	$\mathbf{20_x}$	18_z

Formulas are easily constructed for translating any numeral in a system with one base to a numeral in another base that stands for the same positive integer, Thus we can establish that $10_X = 110_2 = 101_3$ while $1000_2 = 22_3 = 8_{X.}$ Algorithms can be produced for adding or multiplying mixed numerals drawn from different arabic systems. However, algorithms are simplest if all numerals are expressed in the same system.

The following equalities hold no matter what base is used: Let ***b*** be a variable which can take any positive integer as its value. In each ***b***-system of arabic notation.

10_b = ***b*** to the first power 10_2 = (oo) = (oo) = 2_X
100_b = ***b*** to the second power 100_2 = ((oo)(oo)) = (oooo) = 4_X
1000_b = ***b*** to the third power 1000_2 = (((oo)(oo))((oo)(oo))) = (oooooooo) = 8_X
10_3 = (ooo) = (ooo) = 3_X
100_3 = ((ooo)(ooo)(ooo)) = (ooooooooo) = 9_X
1000_3 = (((ooo)(ooo)(ooo))((ooo)(ooo)(ooo))((ooo)(ooo)(ooo)))
= (ooooooooooooooooooooooooooo) = 27_X
Etc....

Certain arithmetic equations hold for all arabic numeral systems. For example, ten times ten always *equals* one hundred, and ten times one hundred always *equals* one thousand even though 'ten' and 'one hundred' stand for different positive integers in each arabic numeral system.

Algorithms in any arabic numeral system are based, as in the decimal system, on basic addition and multiplication tables. In the systems with bases 2 and 3 these tables look like this:

+	1	10
1	10	11
10	11	100

×	1	10
1	1	10
10	10	100

+	1	2	10
1	2	10	11
2	10	11	12
10	11	12	20

×	1	2	10
1	1	2	10
2	2	11	20
10	10	20	100

Algorithms for adding and multiplying larger numbers in each of these systems, requires memorizing the tables of that system, together with rules for "carrying" over when sums or products exceed 10. Since the addition and multiplication tables are different in each system, algorithms differ in different systems, although all have equally effective algorithms for multiplication and addition. Further, addition and multiplication tables in all arabic systems have certain features in common. Compare the tables above with those in the decimal system.

+	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	**10**	**11**
2	3	4	5	6	7	8	9	**10**	**11**	12
3	4	5	6	7	8	9	**10**	**11**	12	13
4	5	6	7	8	9	**10**	**11**	12	13	14
5	6	7	8	9	**10**	**11**	12	13	14	15
6	7	8	9	**10**	**11**	12	13	14	15	16
7	8	9	**10**	**11**	12	13	14	15	16	17
8	9	**10**	**11**	12	13	14	15	16	17	18
9	**10**	**11**	12	13	14	15	16	17	18	19
10	**11**	12	13	14	15	16	17	18	19	20

×	1	2	3	4	5	6	7	8	9	10
1	**1**	2	3	4	5	6	7	8	9	**10**
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	**10**	20	30	40	50	60	70	80	90	**100**

In every arabic numeral system the following rules hold

1) **The multiplication table** has 1, 10, 100, 10 in its four corners, respectively.
2) The lower-left to upper dight diagonal of **the addition table** has 10s, with 11's on their right.
3) In every arabic notation system the laws of commutation, association and the distribution of multiplication over addition hold:

A) (a + b) = (b + a) B) (a × b) = (b × a)
C) (a + (b + c)) = ((a + b) + c) D) (a × (b × c)) = ((a × b) × c)
E) (a × (b + c)) = ((a × b) + (a + c))

4) In every arabic notation system laws of exponents hold for all arabic numerals with only 1 and 0.as digits;
A) Multiplication: $a^n \times a^m = a^{n+m}$
B) Division: $a^n \div a^m = a^{n-m}$, if m> n
C) Raising to a power: $(a^m)^n = a^{n \times m}$
D) Power of a product: $(ab)^n = a^n b^n$

3.3 Digression #1 - Arabic Notation for Non-Integral Bases (e.g., *e*)

In mathematics some arabic notation systems are based on non-integers such as fractions, or numbers such as Euler's ***e*** which begins 2.718281828... and expands indefinitely. We will deal with so-called irrational numbers, and what we can do with them, later from a different point of view.

3.4 Digression #2 - Arabic Numeral Systems and Metric Systems for Physics

Physics deals with things and events in space and time. It associates numbers with distances, areas, volumes and periods of time. In all of these cases, its assumes the things it talks about come in quantities that can be viewed as multiples of units that are equal to each other. The metric system for measuring lengths, widths etc, is based on the decimal system with X or [ooooooooo] as its base. 10 mm = 1 cm, 10cm = 1 decm, 100cm = 1 meter, 1000m = 1km These are all different kinds of spatial lengths. 10 things in a garbage bag = 1garbage bag of things. 10 garbage bags with 10 things each in them, = 100 things in 10 garbage bags. Take away 7 things from one of the bags, you have 93 things in 10 garbage bags. Take away 7 cm from **one meter** of 10 have 93 **cms.** left. I.e., there is a heirarchy of metric distances, each kind of metric distance is a fixed fraction of the kind above it.

"Things" is not part of a hierarchy of kinds. Take away 7 **things** from 100 **things** and you have 93 **things** left, but not 93 things of some kind that are all the same. As from 1 **meter**, we shift to 93 **centimeters**. A **centimeter** is a different kind of thing than a **meter**, but **one meter** is *geometrically* equal to **100 centimeters** by definition**,** so the whole consisting of 100 centimeters is a kind of thing, namely a meter, which is a different kind of thing that its parts, centimeters. The 93 things left are all things of the same kind of thing, in that they are all centimeters (all geometrically equal in size), and they are different kinds of things than the whole meter which existed before 7 centimeters were removed. Taking seven **things** away from one of the garbag bags, and what is left are 93 **things.** But the whole set of 100 things that were in the different garbage bags, was not some kind of thing (like a meter) such that things left are all things of a different kind than the thing that was the whole. Natural numbers assume **individual entities (or things)**, and **groups of individual entities, and groups of groups** of individual things.. But these groups are not at each level associated with a different kind of thing, or a separate kind of individual entity.

To be sure in Arabic notation, a *group of 100* things is equal to 10 *groups of 10* things (or 4 *groups of 25* things or 5 *groups of 20* things, or 50 *groups of 2* things or 2 *groups of 50* things). This make arabic notation in the decimal system seem quite parallel to metric systems based on the decimal system. But a meter, or meter stick, is a physical object, fixed in the physical world and natural objects, distances, etc., are measured by meter sticks, which are marked off into centimeters and millimeters. The currency of the United States also is parallel to the decimal system, with 10 cents worth 1 dime and 10 dimes worth 1 dollar, and $10 bills and $100 bills and $1,000 bills, with equalities, by definition, between these kinds of bills and coins.

Chapter 4 - RATIOS OF NATURAL NUMBERS

The study of natural numbers includes not only the relation of greater and smaller between two numbers, but the *proportions* of greatness or smallness between two numbers. Since every natural number is equal to just one positive integer, we may confine our discussion to the ratios of positive integers. The ratio of any two complex numbers will be equal to the ratio of the two positive integers that are equal to them.

To say positive integer ***a*** is greater than positive integer ***b***, doesn't say much about their size relationship. By addition and subtraction we can tell ***how much greater a*** is than ***b*** ; this is precise additional information. In some cases we can tell precisely ***how many times greater a*** is than ***b*;** this adds more information, but works exactly (without remainder) only when ***a*** is a multiple of ***b.***

To get the precise relationship of relative greatness or smallness, we need to present both numbers in a way that shows the ***ratio of a*** to ***b.*** We use the colon to indicate this. '(***a:b***)' means or stands for the ***ratio of a to b.*** To talk about the ratio of 2 to 3, we use '(2:3)' as the name of that ratio. To speak of the ratio of 2 to 3, does not imply that 2 is divisible by 3, for it is not. In this book '(2:3)' does not mean the same as '2/3' which would imply (incorrectly) that the natural number 2 can be composed of 3 equal positive integers.

If (3×2) = (6×1) then (3:1) = (6:2).

In general if (x×y) = (z×w) then (x:w) = (z:y).

4.1 Ratios

Every particular integer has a positive ratio, ***a:a***, to itself., which we will call a **1-1** ratio**.** For any two different positive integers ***a*** and ***b*** there are two ratios, ***a:b*** and ***b:a***. If ***a*** is *greater than* ***b,*** then ***a:b*** is a *greater ratio* or G-ratio; if ***a*** *is less than* ***b,*** then ***a:b*** is a *lesser ratio* or *L-ratio*. Any L-ratio is the reciprocal of a G-ratio and vice versa The following table of ratios begins the enumeration of ratios in a systematic way. The darkly shaded diagonal contains only **1-1-** ratios. The lightly shaded upper-right section contains only **G-ratios -** ratios that are

1:1 9	2:1 b	3:1 b	4:1 b	5:1 b	6:1 b	7:1 b	8:1 b	9:1 b	10:1 b
1:2 6	2:2 8	3:2 8	4:2 8	5:2 8	6:2 8	7:2 8	8:2 8	9:2 8	10:2 8
1:3 6	2:3 6	3:3 8	4:3 8	5:3 8	6:3 8	7:3 8	8:3 8	9:3 8	10:3 8
1:4 6	2:4 6	3:4 6	4:4 8	5:4 8	6:4 8	7:4 8	8:4 8	9:4 8	10:4 8
1:5 6	2:5 6	3:5 6	4:5 6	5:5 8	6:5 8	7:5 8	8:5 8	9:5 8	10:5 8
1:6 6	2:6 6	3:6 6	4:6 6	5:6 6	6:6 8	7:6 8	8:6 8	9:6 8	10:6 8
1:7 6	2:7 6	3:7 6	4:7 6	5:7 6	6:7 6	7:7 8	8:7 8	9:7 8	10:7 8
1:8 6	2:8 6	3:8 6	4:8 6	5:8 6	6:8 6	7:8 6	8:8 8	9:8 8	10:8 8
1:9 6	2:9 6	3:9 6	4:9 6	5:9 6	6:9 6	7:9 6	8:9 6	9:9 8	10:9 8
1:10 6	2:10 6	3:10 6	4:10 6	5:10 6	6:1 6	7:10 6	8:10 6	9:10 6	10:108

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greater than **1-1**. The lower left unshaded section contains only **L-ratios** - ratios that are less than **1-1**, where (x:y)<(z:w) if and only if (x × w)<(z × y).

The enumeration begins with the ratio of (o) to (o) or 1:1. Its next step is to the L-ratio 1:2 or (o):(oo). The third step is to 2:2 or (oo):(oo) and the fourth step is to 2:1 or (oo):(o), from which it moves to 1**:**3 or (o)**:**(ooo). Thereafter it follows the following form**:** a) after the **$(n-1)^2$** th step the numbers from from 1 to **n** are entered sequentially on the left of '**(...:n)**' b) after **(n:n)** the numbers from **n-1** to **1** are entered in reverse order on the right of '**(n:...)'.** The last step is the **n^{2rd}** step in the total sequence, so one proceeds in the same fashion with **n+1.**

Step	Ratios	
1 =1^2	(1:1)	
2	(1:2)	
3	(2:2)	
4 =2^2	(2:1)	
5	**(1:3)**	**(1:n)**
6	**(2:3)**	**(...:n)**
7	**(3:3)**	**(n.:n)**
8	**(3:2)**	**(n:..)**
9 =3^2	**(3:1)**	**(n:1)**

4.2 Proportional Equality, vs. Numerical Equality

Ratios are not equal to natural numbers. Given our definition of equality - "reducible to the same positive integer" - ratios are pairs of natural numbers and can not be reduced to a single positive integer.

This is clear in primitive notation (1×2) is,by definition, ((oo)), and by our defintion of equality (1×2) = 2 i.e., ((oo)) = (oo), and (1×7) = 7, i.e., ((ooooooo)) = (ooooooo), and so on for all positive integers.

But the concept of division in natural numbers is different than the concept of a ratio. The former is conditioned on multiplication. It does not hold of most pairs of positive integers: 7 is not divisible by 2, or by 3, or by 4. But all pairs of natural numbers have specific ratios.The customary equations 2/1 = 2 or in general (n/1) = n are okay. They follow from our definition of division.?? For (1×2) = 2 and in general (1×n) = n.; it follows that n/1 = n,

Pure number theory does not preuppose a metric system. Metric systems assume that that the ratio (14**:**2) is equal to 7, and in general, If (x × y) = z, then **x =**(y**:**z**)** . We do not say this, because a positive integer can not be equal (on our definition) to a ratio. They are objects of different kinds. We may say that if If (x × y) = z, then z is divisible by x, ot by y, or that If (x×y) = z, then x is a factor (a possible divisor) of z. But this is not so say that is equal to z**:**x. Rather if (x×y) = z then z has the relational propertiy of being equal to the sum of y x's.

No ratios of natural numbers are equal to a positive integer or to a natural numbers. But there is a kind of equality, different from the equality of natural numbers, that they may have. We will call this *proportional equality* and symbolize it by ' $_p$=.' It holds only between ratios, not be tween positive integers or natural numbers..We will say that ***a*:*b*** $_p$= ***c*:*d*** if and only if ***a***×***d*** = ***b***×***c***. Thus, for example, **2:3** $_p$**= 4:6** ***,*** because **(2 × 6) = (3 × 4) .**

Df '$_p$=': [((â : å) $_p$= (ã: ä)) syn$_{df}$ ((â × ã) = (ä × å))]

The defined synonymy of the two whole expressions does not entail that components of the two different kinds of equality are synonymous.or equal. '**(â × ã)** ' is not equal to, or synonymous with, or intersubstitutable with **'(â :ä)**' ; the instance of the former, (**2:3),** does not bear these relations to **(2 × 6)** as instance of the latter**.** In the case of proportional equality, many ratios will be proportionately equal to each other. This differs from natural-number-equality. When any two natural numbers are equal, they are reducible to one only one positive integer. But ratios each are composed of two independent positive integers, and two independent entities can not be reduced to just one positive integer.

4.3 Greater and Lesser Ratios

What does it mean to say that one ratio is greater, or lesser proportion wise than another?

Df '$_p$>': [((â :ä) $_p$> (å :ã)) syn$_{df}$ ((â × ã) > (ä × å))]
Df '$_p$<': [((â :ä) $_p$< (å :ã)) syn$_{df}$ ((â × ã) < (ä × å))]

Consider the ratios **(1:3)**, **(3:3)**, **(3:1);** they are proportionally greater or less. like **1/3, 1/1, or 3/1.**

T((3×3) > (3×1)) .: T((3:3) $_p$> (1:3)) **T((3×1) < (3×3)) .: T((3:3) $_p$< (3:1))**
T((3×3) > (1×3)) .: T((3:1) $_p$> (3:3)) **T((1×3) < (3×3)) .: T((1:3) $_p$< (3:3))**
T((3×3) > (1×1)) .: T((3:1) $_p$> (1:3)) **T ((1×1) < (3×3)) .: T((1:3) $_p$< (3:1))**

Or consider the question, "Is **(13:57)** greater than **(5:23)?**" Since by Df '$_p$>',

((13:57) $_p$> (5:23)) syn$_{df}$ ((13 × 23) > (57 × 5))

the question becomes, "Is it true that **((13×23) > (57×5))?"** Which, because **(13×23)=299** and **(57× 5)= 285,** becomes "Is it true that **299 > 285?**" To which the answer is obviously, **yes.**

The system of enumeration in Section 4.1 is such that for any ***n***, the sequence from the (***n***^{2}, ***2(n* - 1)**)th to the ***n***^{2} terms in the sequence are arranged in order of size. For example, the 5th (=(**3**^{2}□ 2(**3**□ 1)) to the 9th (=**3**^{2}) terms are (1:3), (2:3), (3:3), (3:2), (3:1) **;**
the 10th (=(**4**^{2}□ 2(**4**□1)) to the 16th (=**4**^{2}) terms are (1:4), (2:4), (3:4), (4:4), (4:3), (4:2), (4:1)**.**

4.4 Decimal Points Addition of Ratios

It makes sense 1) to *multiple* or divide a ratio by a natural number, and 2) to *multiply* one ratio of natural numbers by another; and 3) to find the ratio that one ratio of natural numbers bears to another. But the concept of *adding* one ratio to another requires assumptions that are required for some application, but not for all applications.

Multiplying Ratios. Ratios can be multiplied by numbers greater than 1**;** the result is a ratio greater than the initial ratiio:

1) ((x × 2)**:**3)>(2**:**3)

When both members of the ratio are multiplied by the same number, the result is proportionately equal to the initial ratio

2) ((x × 2)**:**(x × 3)) $_p$= (2**:**3)

When the denominator of the ratio is multiplied by a number greater than 1, the resulting ratio is less than the original ratio. This may be viewed as dividing the ratio.:

3) (2**:**(x × 3)) < (2**:**3)

A ratio can also be multiplied by another ratio. The phrase "one half of three halves is three fourths" is expressed, using multiplication, as ***(1:2)*×*(3:2) = (3:4),*** and is gotten by the formula, ***((a:b)*×*(c:d)) = ((a*×*c):(b*×*d)),*** For example,

4) ((1**:**2*)*×(3**:**2)) = ((1×3)**:**(2×2)) **=** (3**:**4)

The Ratios of Ratios. There is such a thing as the ratio of one ratio to another ratio. Not only can one ratio be greater, less than or equal proportionately to another**;** they may stand in definite ratios to each other. Thus the ratio (3**:**2) is three times (1**:**2)**;**.in symbolism, (3**:**2)**:**(1**:**2) $_p$= (3**:**1). The ratio (1**:**2) is one third of the ratio (3**:**2), i.e., (1**:**2)**:**(3**:**2) $_p$= (1**:**3). And since (1**:**2) $_p$= (5**:**10)**,** (5**:**10) is one third of (3**:**2) i.e., (3**:**2)**:**(5**:**10) $_p$= (3**:**1). The formula is:

5) (((a**:**b)**:**(c**:**d)) $_p$= ((a × d)**:**(b × c))

The relation of being a ratio is like the relation of being $_p$=.. **a:*b*** $_p$= ***c:d*** syn ***a***×***d*** = ***b***×***c***.
Examples**:** ((1**:**2)**:**(3**:**2)) $_p$= ((1 × 2)**:**(2 × 3)) $_p$= (2**:**6) $_p$= (1**:**3)
((3**:**2)**:**(1**:**2)) $_p$= ((3 × 2)**:**(2 × 1)) $_p$= (6**:**2) $_p$= (3**:**1)
((5**:**10)**:**(3**:**2)) $_p$= ((5 × 2)**:**(10 × 3)) $_p$= (10**:**30) $_p$= (1**:**3)

Note that this is not the same as multiplying ratios.
Multiplying two ratios is: ***((a:b)×(c:d)) = ((a×c):(b×d))*** e.g., (1:2)×(3:2) = (3:4)
Finding the ratio of two ratios is ***(((a:b):(c:d))*** $_p$***= ((a×d):(b×c))*** e.g., (1:2) **:**(3:2) = (1:3)

Addition of Ratios.

1. Suppose I have two grocery bags. Bag A has 3 things in it, Bag B has 7 things in it. The ratios of things in the two bags are (3:7) or (7:3), i.e., Bag B's contents are less than Bag A's in the proportion (3:7) or Bag A's contents are greater s than Bag B's in the proportion (7:3)**.**

2. Now suppose I have two bags, Bag C and Bag D, and that Bag C has 13 things in it while Bag D has only 2. The ratios of things in the two bags are (2:13) or (13:2), i.e., Bag D's contents are less than Bag C's in the proportion (2:13) or Bag C's contents are greater s than Bag D's in the proportion (13:2)**.**

3. What would it mean here to add the ratio **(3:7)** to the ratio **(2:13)**? What would **'(3:7)+(13:2)'** mean**?** What would **'(7:3)+(2:13)'** mean**?** Does the result of adding a**:**b to c**:**d equal (a+c)**:**(b+d) e.g. does **(3:7)+(13:2) = (16:9)** analogously to multiplication? If not, why not? And which of the ratios should be added, any way, since each has two versions?

In standard mathematics the addition of fractions is done by the following formula:

(a/b) + (c/d) = (((a × d) +(b × c))/(b × d))

Replacing slash marks by colons, this becomes

(a**:**b) + (c**:**d) = (((a × d) +(b × c))**:**(b × d))

Thus for example: **(3:7)+(2:13) = (((3×13) + (7× 2)):(7×13)) = (39 + 14):91 = (53:91)**
Or (3**:**8)+(2**:**12) = (((3×12) + (8× 2))**:**(8×12)) = (36 + 16)**:**96 = (52**:**96) $_p$= (13**:**24)
Or (2**:**8)+(3**:**12) = (((2×12) + (8× 3))**:**(8×12)) = (24 + 24)**:**96 = (48**:**96) $_p$= (1**:**2)
Or (1**:**4)+(1**:**4) = (((1 × 4) + (4× 1))**:**(4 × 4)) = (4 + 4)**:**16 = (8 **:** 16) $_p$= (1**:**2)

Why is the answer **(3:7)+(13:2)** = **(53:91)** better than **(3:7)+(13:2)** = **(16:9)**? Well, if you assume that all units are entities equal in a certain respect (physical size, weight, units of time), then we can find applications which produces visibly confirmable sums of these. But this is an assumption that goes beyond the definition of pure numbers or pure ratios of numbers. That is, it doesn't follow from pure number theory.

The followin theorem which lies at the base of trignometry, is of particular interest. Under cerrtain conditions,

If $x < y < z$ and $(x + y) \geq z$, then $\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 = \left(\frac{z}{z}\right)^2$

or, in our notation, If $x < y < z$ and $(x + y) \geq z$, then $(x{:}z)^2 + (y{:}z)^2 = (z{:}z)^2$

Reverting to the fractional notation, the proof of this begins with

If $x < y < z$ and $(x + y) \geq z$, then $\frac{\frac{x}{z}}{\frac{z}{x}} + \frac{\frac{y}{z}}{\frac{z}{y}} = \frac{\frac{z}{z}}{\frac{z}{z}}$

From this is follows that

If $x < y < z$ and $(x + y) \geq z$, then $\left(\frac{x \times x}{z}\right) + \left(\frac{y \times y}{z}\right) = \left(\frac{z \times z}{z}\right)$

and hence, If $x < y < z$ and $(x + y) \geq z$, then $\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 = \left(\frac{z}{z}\right)^2$

x y z

If $x < y < z$ and $(x + y) \geq z$, then $\frac{z}{\frac{z}{x}} + \frac{z}{\frac{z}{y}} = \frac{z}{\frac{z}{z}} = \frac{z}{z} = \frac{1}{1}$

By addition of ratios of ratios, we get

If $x < y < z$ and $(x + y) \geq z$, then $\frac{\frac{x}{z}}{\frac{z}{x}} + \frac{\frac{y}{z}}{\frac{z}{y}} = \frac{\frac{x}{z}\times\frac{z}{y} + \frac{y}{z}\times\frac{z}{x}}{\frac{z\times z}{x\times y}} = \frac{\frac{x+y}{y\ x}}{\frac{z^2}{x\times y}} = \frac{\frac{x^2+y^2}{x\times y}}{\frac{z^2}{x\times y}} = \frac{x^2+y^2}{z^2} = 1$

When does $\frac{x^2+y^2}{z^2} = 1$? When and only when $x^2+y^2 = z^2\times 1$.

Under what conditions does this hold?

In geometry ratios are associated with comparative lengths of lines, or with ratios of areas. The Pythagorean Theorem proved that in plane Euclidean geometry the two square areas built on the sides of a right triangle would equal in total area the square built on the hypotenuse.

In standard mathematics is is taken for granted that the ratio (n**:**1) = n, where n is a positive integer or a natural number as we have defined it. 76/1 = 76. While a ratis not identical with any natural number, it true that (1xn) = 1. However, proportional equality is preserved in ratios if both the numerator and the denominator of the ratio are multiplied by the same numbers. Thus (76:1) $_{p}$= ((10x76)**:**(10:1))

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3.4 Irrationals as Approximate Ratios relative to a base.

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Finding proximate square roots of Positive Integers:

Df '$_{p}$=': [((â : å) $_{p}$= (ã : ä)) syn$_{df}$ ((â × ã) = (ä × å))]
Df '$_{p}$>': [((â : ä) $_{p}$> (å : ã)) syn$_{df}$ ((â × ã) > (ä × å))]
Df '$_{p}$<': [((â : ä) $_{p}$< (å : ã)) syn$_{df}$ ((â × ã) < (ä × å))]

"(x/d) is the *proximate proportional equal* of (a/b), given the base d"

((x/d) $_{PRXp}$= (a/b)) & ~(Ey)(PIy & ((y/d) $_{PRXp}$= (a/b)) & ~ y=x))

means

"(x/d) is the rational fraction with the denominator d that is *closest to being* $_{p}$*equal* to (a/b) & no Positive Integer is such that it, with the denominator d, is closer to being $_{p}$equal to (a/b)"

If PIy & PI z & PIw) ***then*** (Ex)(PIx & ((x/y) $_{PRXp}$= (z/w))
& (v)(((v/t) $_{PRXp}$= (z/w)) => v=x))

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2.5. Ratios' Relations and Properties (Rational Numbers)

If we wish to assert that two numbers stand in a certain relation, we present them as an ordered pair, (n_1,n_2) this ordered pair becomes the subjects of a predicate. E.g., "Is greater than", "is Less than", "is integrally equal to", "is a factor of...". In English we usually put these predicates between the two subject terms as in "n_1 is greater than n_2", "n_1 is Less than n_2", "n_1 is integrally equal to n_2", etc., and this is reflected in the conventional mathematical language**:** (n_1

$>n_2)$, $(n_1<n_2)$, $(n_1=n_2)$.. Because mathematics has many predicates that apply to ordered triples, ordered, n-tuples, and we want to regularize the form of predicates with the predicates preceding the subjects., we might use the notation $>(n_1,n_2)$, $<(n_1,n_2)$, $=(n_1,n_2)$., but we shall avois this untill it becomes necessary.

In many cases we want to talk about the relations between two or more ordered pairs of numbers, rather than about the relation between members of one ordered pair of natural numbers.

The statements "2 is less than 4" and the "3 is less than 6", each talk about a relation between two numbers in an ordered pair. But we may wish to point out that the ordered pairs (2,4) and (3,6) have a cetain relationship, e.g., their two members stand in the same proportional relation as (1,2). When we take ordered pairs of natural numbers as subject terms instatements about relations between them, we call them ratios, and ratios of of numbers are expressed as $(n_1:n_2)$ Thus we may say that "The ratio of (1:2) is similar to the ratio of (3:6) in the proportions their members bear to one another." One kind of relation between ratios, let us call it "proportional equality" or '$_P$=', can be defined rigorously using integral equality at the base, .

There are six possibilities, with x first in the antecedent and x first in the consequent:

xyzw => xyzw, xywz, <u>xzyw</u>, xzwy, xwyz, <u>xwzy</u>

Two preserve proportionality: xz<u>yw</u> and xw<u>zy</u>

Two fail because the second ratio is the inverse of the good cases second ratio: xz<u>wy</u> & xw<u>yz</u>

The other two, <u>xy</u>zw => <u>xy</u>zw, <u>xy</u>wz, fail because the second ratio is the inverse of the first???

If x × y = z ×w then x = (z × w):y) and **then x:z = w:y or x:w = z:y**

xzwy **xwzy**

If x × y = z ×w then (x:w = z:y) xyzw, **xwzy**

[E.g., **if** 3×20 = 4×15 **then** (3:15 = 4:20)] or 3/15 = 4/20 = 1/5 = 0.2000
if 20×3 = 4×15 **then** (20:15= 4:3)] or 20/15 = 4/3 = 4/3 =1.3333
if 3×20 = 15×4 **then** (3:4 = 15:20)] or 3/4 = 15/20 = 3/4 = 0.7500
if 20×3 = 15×4 **then** (20:4 = 15:3] or 20/4 = 15/3 = 5/1= 5.0000
if 4×15 = 3×20 **then** (4:20 = 3:15)] or 4/20 = 3/15 = 1/5
if 15×4 = 3×20 **then** (15:20 =3:4)] or 15/20 = 3/4 = 3/4
if 4×15 = 20×3 **then** (4:3 = 20:15)] or 4/3 = 20/15 = 4/3
if 15×4 = 20×3 **then** (15:3 = 20:4)] or 15/3 = 20/4 = 5/1

Or alternative is **OK** **If x × y = z × w then (x:z = w:y)** xyzw, **xzwy**

[E.g., **if** 3×20 = 4×15 **then** (3:4 = 15:20)] or 3/4 = 15/20 = 3/4 = 0.7500
if 20×3 = 4×15 **then** (20:4 = 15:3)] or 20/4 = 15/3 = 5/1 = 5.0000
if 3×20 = 15×4 **then** (3:15 = 4:20)] or 3/15 = 4/20 = 1/5 = 0.2000
if 20×3 = 15×4 **then** (20:15 = 4:3] or 20/15 = 4/3 = 4/3 = 1.3333
if 4×15 = 3×20 **then** (4:3 = 20:15)] or 4/3 = 20/15 = 1/5
if 15×4 = 3×20 **then** (15:3 =20:4)] or 15/3 = 20/4 = 5/1
if 4×15 = 20×3 **then** (4:20 = 3:15)] or 4/20 = 3/15 = 1/5
if 15×4 = 20×3 **then** (15:20 = 30:4)] or 15/20 = 3/4 = 4/3

<u>Not</u> **If x × y = z × w then (x:y = z:w)** **xyzw, xyzw**
if 20×3 = 15× 4 **then** (20:3 ≠ 15:4) 6.666 ≠ 3.75

<u>Not</u> **If x × y = z × w then (x:y** = **w:z)** **xyzw, xywz**
if 20×3 = 15×4 **then** (20:3 ≠ 4:15) 6.666 ≠ 2.666

<u>Not</u> **If x × y = z × w then (x:z = y:w)** **xyzw, xzyw**

if 20×3 = 15×4 **then** (20**:**15 ≠ 3**:**4) 1.333 ≠ .75
Not **If x × y = z × w then (x:w = y:z)** **xyzw, xwyz**
if 20×3 = 15×4 **then** (20**:**4 ≠ 3**:**15) 5.000 ≠ .2000

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Finding proximate square roots of Positive Integers:
Df. [(x/d) $_{PRX}$= $(a/b)] Syn [(x/d)^2 $_{PRX}$= (a/b)]
Theorem: (y)(z)(w)(***If*** PIy & PI z & PIw)
then (Ex)(PIx & (x/y)^2 $_{PRXp}$= (z/w) & (v)(((v/t)^2 $_{PRXp}$= (z/w)) => v=x))

Problem**:** Instantiate y,z,w with positive integers, then find the value of x
Example:

1. Let y=1,000, z=2, w= 1
2. (***If*** PI(1,000) & PI(2) & PI(1)
then (Ex)(PIx & (x/1,000)^2 $_{PRXp}$= (2/1)
& (v)(((v/1,000)^2 $_{PRXp}$= (2/1)) => v=x))
3. PI(1,000) & PI(2) & PI(1)
4. (Ex)(PIx & (x/1,000)^2 $_{PRXp}$= (2/1) & (v)(((v/1,000)^2 $_{PRXp}$= (2/1)) => v=x))

Problem**: What PI = x?**
(x/1000)^2 $_{PRXp}$= (2/1)
(x^2 / 1000^2) $_{PRXp}$= (2/1)
(x^2 × 1) $_{PRXp}$= (2 × 1000^2)

y=1,000, z=2, w= 1
1. 1,000^2.= 1,000,000
2. (2 × 1,000,000) = 2,000,000
3. (2 × 1,000^2.) = 2,000,000
4. 1,414^2 = 1,999,396
5. |1,999,396□ 2,000,000| = 604
6. |1,414^2□ (2 × 1,000^2.)| = 604 | (w × x^2)□ (z × y^2)| = 604
7. 1,413 = (1,414 □ 1)
8. 1,413^2 = 1,996,569
9. (1,414 □ 1)^2 = 1,996,569
10. |1,996,569 □ 2,000,000| = 3,431
11. |(1,414□1)^2□ (2×1,000^2.)| = 3,431 |(w × (x -1)^2)□ (z× y^2)| = 3.431
12. 1,415 = (1,414 + 1)
13. (1,414 + 1)^2 = 2,002,225
14 .|2,002,225□ 2,000,000| = 2,225
15 |(1,414 +1)^2 □ (2×1,000^2.)| = 2,225 | (w × (x +1)^2)□ (z × y^2)| = 2,225
16. 3,431 ≤ 604 ≤ 2,225
17. |(1,414□1)^2□ (2×1,000^2.)| ≤ |1,414^2□ (2× ,000^2.)| ≤ |(1,414 +1)^2 □ (2×1,000^2.)|
|(w × (x -1)^2)□ (z× y^2)| ≤ | (w × x^2)□ (z × y^2)| ≤ |(w × (x -1)^2)□ (z× y^2)|

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"The proximate square root of (Î **:** Ï) for the base Ñ is (Ð**:**Ñ)"
means

"The rational fraction that is closest to (Î / Ï) with the denominator Ñ is (Ð/Ñ)"

"The *proximate square root* of (a:b) for the base d is (c:d)" ($_{PX}$@2 ,d) = (c/d))
means
"The rational fraction that is closest to (a/b) with the denominator d is (c/d)"

"The *proximate equal* of (a/b) for the base d is (c/d)" (@2 ,d) $_{PRX}$= (c/d))
means
"The rational fraction that is *closest to being equal* to (a/b) with the denominator d, is (c/d)"

Form of the Problem: What is the most exact square root of x:1, relative to the denominator y ?
Form of the Answer: The most exact square root of x:1, relative to the denominator y = z .

To state an actual problem, is to replace x and y by positive integers.

E.g., What is the most exact square root of 2:1, relative to the denominator 5 ?

To state an actual answer, is to find a positive integer to replace z, such that it can be proved that the form of the answer is true when x and y are replaced by the positive integers as stated in the actual problem.

E.g., The most exact square root of 2:1, relative to the denominator 5 = 7/5.

What is the most exact square root of 2:1, relative to the denominator n ?
Problem: (@2 ; 5) _ (?/5)
Amswer: (@2 ;5) _ (7;5)
Proof of answer:
"the closest approximation relative to denominator 5, for the square root of 2/1, is 7/5"
Proof: 1) (7/5)×(7/5) = $(7/5)^2$ = (49/25)
2) 49/25 is the closest proportional to 2/1. relative to the denominator 5
3) ((49×1)_ (2×25))
For, $|2(5□1)^2 □ (7^2)| \leq |2(5)^2 □ (7^2)| \leq |2(5+1)^2 □ (7^2)|)$
$|2(4)^2 □ (7^2)| \leq |2(5)^2 □ (7^2)| \leq |2(6)^2 □ (7^2)|)$
|32 □ 49)| ≤ |50 □ 49| ≤ |72 □ 49|)
17 ≥ 1 ≤ 23

"The proximate square root of 2/1 with the denominator 1,000, is 1,414/1,000"
Proof: 1) (1414/1000)×(1414/1000) = $(1414/1000)^2$ = (1,999,396/1,000,000)
2) (1,999,396/1,000,000) is the closest proportional to 2/1. relative to the base 1.000
3) ((1,999,396/1,000,000) $_{PRX}$= (2/1)
4) ((1,999,396 × 1) $_{PRXP}$= (1,000,000 × 2)
5) ((1,999,396) $_{PRXP}$= (2,000,000)
$|(1414□1)^2 □ (2\times 1{,}000^2)| \leq |(1414)^2 □ (2\times 1{,}000^2)| \leq |(1414+1)^2 □ (2\times1{,}000^2))$
$|(1413)^2 □ (2\times 1{,}000^2)| \leq |(1414)^2 □ (2\times 1{,}000^2)| \leq |(1415)^2 □ (2\times1{,}000^2))$
|1,996,569□ 2,000,000| ≤ |(1,999,396 □ 2,000,000| ≤ |2,002,225□ 2,000,000))
3,431 ≤ 604 ≤ 2,225

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"(x/d) is the *proximate equal* of (a/b), given the base d" (x/d) $_{PRX}$= (a/b))

means
"(x/d) is the rational fraction with the denominator d that is *closest to being equal* to (a/b)"

Df. [(x/d) $_{PRX}$= \$(a/b)] Syn [$(x/d)^2$ $_{PRX}$= (a/b)]
Theorem: (y)(z)(w)(***If*** PIy & PI z & PIw)
then (Ex)(PIx & $(x/y)^2$ $_{PRXp}$= (z/w) & (v)((($(v/t)^2$ $_{PRXp}$= (z/w)) => v=x))

Problem**:** Instantiate y,z,w with positive integers, then find the value of x
Example:

1. Let y=1,000, z=2, w= 1
2. (***If*** PI(1,000) & PI(2) & PI(1)
then (Ex)(PIx & $(x/1{,}000)^2$ $_{PRXp}$= (2/1)
& (v)((($(v/1{,}000)^2$ $_{PRXp}$= (2/1)) => v=x))
3. PI(1,000) & PI(2) & PI(1)
4. (Ex)(PIx & $(x/1{,}000)^2$ $_{PRXp}$= (2/1) & (v)((($(v/1{,}000)^2$ $_{PRXp}$= (2/1)) => v=x))

Problem**: What PI = x?**
$(x/1000)^2$ $_{PRXp}$= (2/1)
$(x^2 / 1000^2)$ $_{PRXp}$= (2/1)
$(x^2 \times 1)$ $_{PRXp}$= (2×1000^2)

y=1,000, z=2, w= 1
1. $1{,}000^2$.= 1,000,000
2. $(2 \times 1{,}000{,}000) = 2{,}000{,}000$
3. $(2 \times 1{,}000^2.) = 2{,}000{,}000$
4. $1{,}414^2 = 1{,}999{,}396$
5. $|1{,}999{,}396 \square\ 2{,}000{,}000| = 604$
6. $|1{,}414^2 \square\ (2 \times 1{,}000^2.)| = 604$ $|(w \times x^2) \square\ (z \times y^2)| = 604$
7. $1{,}413 = (1{,}414 \square\ 1)$
8. $1{,}413^2 = 1{,}996{,}569$
9. $(1{,}414 \square\ 1)^2 = 1{,}996{,}569$
10. $|1{,}996{,}569 \square\ 2{,}000{,}000| = 3{,}431$
11. $|(1{,}414 \square 1)^2 \square\ (2 \times 1{,}000^2.)| = 3{,}431$ $|(w \times (x-1)^2) \square\ (z \times y^2)| = 3.431$
12. $1{,}415 = (1{,}414 + 1)$
13. $(1{,}414 + 1)^2 = 2{,}002{,}225$

14 .$|2{,}002{,}225 \square\ 2{,}000{,}000| = 2{,}225$
15 $|(1{,}414 + 1)^2 \square\ (2 \times 1{,}000^2.)| = 2{,}225$ $|(w \times (x+1)^2) \square\ (z \times y^2)| = 2{,}225$

16. $3{,}431 \leq 604 \leq 2{,}225$
17. $|(1{,}414 \square 1)^2 \square\ (2 \times 1{,}000^2.)| \leq |1{,}414^2 \square\ (2 \times ,000^2.)| \leq |(1{,}414 + 1)^2 \square\ (2 \times 1{,}000^2.)|$
$|(w \times (x-1)^2) \square\ (z \times y^2)| \leq |(w \times x^2) \square\ (z \times y^2)| \leq |(w \times (x-1)^2) \square\ (z \times y^2)|$

==

The rational ratios we are interested in finding are ratios with some denominator n, that are the nearest approximation to satisfying some irrational function.

We start by giving an example. The square root of 13, it is said, is an irrational number. But every answer ever given to the question "what number is the square root of 13 ?" is always rational fraction that solves the problem approximately, but not exactly. There are an unlimited number of correct approximate answers, none of them equal to each other. For example, in the decimal system, 3.6, 3.61, 3.606, 3,6056, 3.60555, ...and infinitely more are all accurately described as *the* approximate square root of 13 relative relative to its denominator. Each not only is approximate to the nearest tenth, one-hundredth, one thousandth, ten-thousandth, and hundred-thousandth (in the decmial system) respectively. So instead of the loose and inaccurate predicate " â is the square root of ã", we shall define the predicate,

"â is the value to the nearest ä of â × â = (ã**:**ä)"

This means â is the ratio of natural numbers with the natural number ä as denominator, that comes closest to satisfies a square root of 13. To characterize this set, we need to meet two conditions. One I call the approximation condition, the other the uniqueness condition.

a) the approximation condition, or delimitation condition, marks off a sub-set of ratios, in effect, at least one of which must be chosen given any specified denominator
b) the uniqueness condition selects just one of the sub-set.approximate ratios, as the nearest ratio

3.41 Example: "Square root"defined

In the case of $y =_z !x$ what we are looking for is some ratio, y**:**z,
such that (y**:**z × y**:**z) is $_p$= to the closest possible rational ratio
to the rational ratio x**:**z.

.

a) The approximation condition

First we choose the denominator, w, and we require (approximation condition)

That:	$(13(w\text{-}1)^2$	<	z^2	<	$13\ (w+1)^2)$
Let w=100	$(13\times(100\text{-}1)^2$	<	z^2	<	$13\times(100+1)^2)$
	$(13\times(99)^2)$	<	z^2	<	$13\times(101)^2$
$13\times(99)^2 = 127{,}413$		<	z^2	<	$132{,}612 = 13\times(101)^2$

if z =						
356	**127,413**	**>**	**126,736**	**#**	**132,613**	
357	127,413	<	127,449	<	132,613	
358	127,413	<	128,164	<	132,613	
359	127,413	<	128,881	<	132,613	Thus the approximation
360	127,413	<	129,600	<	132,613	condition restricts
361	127,413	<	130,321	<	132,613	values of @13 relative
362	127,413	<	131,044	<	132,613	to the denominator 100,
363	127,413	<	131,769	<	132,613	to eight values (on the

364	127,413	<	132,496	<	132,613	left)
365	**133,225**	**<**	**133,225**	**>**	**132,613**	

6) the Uniqueness condition

Next we require that just one of the values in the sub-set above be chosen, and that this be the one whose square is closest to a number reducible to 13/1; with denominator = 100, this turns out to be 361. I.e., the square root of 13, to the nearest 100^{th} = 361/100 or 3.61.

The uniqueness condition in this case is expressed as

$$\backslash (z-1)^2 - 13(100)^2 \backslash > \backslash (z^2 - 13(100)^2 \backslash < \backslash (z+1)^2 - 13(100)2 \backslash$$

That is, the difference between z^2 and 130,000 must be less than both the difference between $(z+1)^2$ and 130,000 (ie. $13(100)^2$). Now the differences involved with respect to different values of z are as follows:

$(357)^2 = 127{,}449$	|127,449 - 130,000 \	= 2551
$(358)^2 = 128{,}164$	|128,164 - 130,000 \	= 1839
$(359)^2 = 128{,}881$	|128,881 - 130,000 \	= 1119
$(360)^2 = 129{,}600$	|129,600 - 130,000 \	= 400
$(361)^2 = 130{,}321$	|130,321 - 130,000 \	= 321 p
$(362)^2 = 131{,}044$	|131,044 - 130,000 \	= 1044
$(363)^2 = 131{,}769$	|131,769 - 130,000 \	= 1769
$(364)^2 = 132{,}496$	|132,496 - 130,000 \	= 2496

To meet the uniqueness condition the difference in the right=hand column must be both less than the one above it and less than the one below it. Obviously this can apply to only one case, in this case, 361, whose square is only 321 different from 130,000. Hence the proximate value of the square root of 13, for the denominator 100 = $\frac{361}{100}$, i.e., in decimal notation, 3.61.

The preceding thus determines the value of 13 for the denominator 100, the set of values for all different denominators, i.e., for every natural number as denominator, is settled in the same way. For any given denominator, w, the approximation condition selects a subset of positive numbers (and through them a sub-set of ratios) of which is

$$Z\ (13 \times \backslash w-1\backslash^2 < z^2 < 13 \times (w+1)^2)$$

and from these z's it selects the one and only z such that

$$\backslash (z-1)^2 - 13w^2 \backslash > \backslash z^2 - 13w^2 \backslash < \backslash (z+1)^2 - 13w^2 \backslash$$

We may therefore define $\#_w 13$ as the set

$$Y\ (Ez)\ [y=\frac{z}{w} . ((13(w-1)^2 < z2 < 13(w+1)^2) . (\backslash(z-1)^2 - 13w^2 > \backslash z^2 -13w^2 \backslash< \backslash(z+1)^2 - 13w2\backslash))]$$

And in general, we may define the function @$_w$a (assuming a is eP) as:
Y(Ez) [y=$\frac{z}{w}$. ((a \ w-1 \$^2$ > z^2 > a(w+1)^2) . (\(z-1)^2 -aw2 \ } \ z2 -aw^2 { \(z+1)^2 -aw^2 \))]
[see what page]

We may if we wish, add a couple of refinements. As it stands, with w ranging over all positive integers presumably, we will get some strange members of the set @$_w$a where w is small. We may wish to restrict ourselves to some sub-set of this set. For example, consider decimal expressions of @13 : $\frac{36}{10}$, $\frac{361}{100}$ $\frac{3606}{1000}$, 10000 $\frac{36056}{}$, $\frac{360555}{100000}$, . . .

or, 3.6, 3.61, 3.605, 3.6056, 3.60555, . . .

this set may be expressed by replacing 'w' with '10^n getting ' %10^n 13. ' as the name of all decimal approximations which we call the square root of 13 (to the denominator w, where w=10^n for some n).

Another refinement which might be added, is a convergence condition. This would require that ratios with larger denominators only be admitted to the set if they are "closer" to an exact fit than any ratio with a smaller denominator. This condition can be expressed by adding the following clause to the definition of $_w$a given above:

(u) (v) ($\frac{u}{v}$ e e@ $_w$a. . (Ez) (Ew) (($\frac{z}{w}$ e@ $_w$a. . V< w) 7 (# $\frac{z^2 - aw^2}{z^2}$# < # $\frac{u^2-av^2}{v^2}$#))

This in effect will eliminate any ratio with a larger denominator which is less close to its successors than some ratio with a smaller denominator. For example,

$\frac{36}{10}^2$ = $\frac{1296}{100}$; it falls $\frac{4}{100}$ (0.04) short of being $\frac{1300}{100}$ (i.e.,.13)

and $\frac{361}{100}^2$ = $\frac{130,321}{10,000}$; it is $\frac{321}{10,000}$ away from being $\frac{130,000}{10,000}$ (i.e., 13)

and $\frac{3606}{1000}$ = $\frac{13,003,236}{10,000,000}$; it is $\frac{3,236}{10,000,000}$ away from $\frac{13,000,000}{10,000,000}$ (i.e., 13)

what we want, by the convergence condition is that each ratio, as the denominators get higher, comes "closer" to the ration 13/1

Now $\frac{321}{10,000}$ Is less of a difference than $\frac{4}{100}$ (.0321 < .04) hence 361/100 is okay.

And $\frac{.0003236}{10,000}$ < .0321 hence 3606/1000 is okay; it converges.

==

Distribution of multiplication over addition preserves equality**:**

(3 x (4 + 5)) = (((oooo)(ooooo))((oooo)(ooooo))((oooo)(ooooo)))
= 27 = (ooooooooooooooooooooooooooo)
= ((3x4) + (3x5)) = (((oooo)(oooo)(oooo))((ooooo)(ooooo)(ooooo))
= 27 = (ooooooooooooooooooooooooooo)

Distribution of addition over multiplication does *not* preserve equality**:**

(3 + (4 x 5)) = ((ooo)((ooooo)(ooooo)(ooooo)(ooooo)))
= 23 = (ooooooooooooooooooooooo)
((3+4) x (3+5)) = (7 x (3+5)) =
(((ooo)(ooooo))((ooo)(ooooo))((ooo)(ooooo))((ooo)(ooooo))((ooo)(ooooo))((ooo)(ooooo))((ooo)(ooooo))) = 56 = (oooooooooooooooooooooooooooooooooooooooooooooooooooooooo)

============

There are many relations of **Similarity and Difference** among abstract groups of entities are ternary, not binary, relations. The two signs '(()()())' and '(()()())' are *similar* ***with respect to*** parenthetical structure but *different* ***with respect to*** location. If we call the first one ***a*** and the second one ***b***, then we may symbolize their similarity and difference as SIM(***a***,***b***,S) and DIFF(***a***,***b***,L).

In contemporary logic and mathematics the term 'identity' is conflated with 'equality' in logic and mathematics. But we distingusih the two relations here. In one sense, it is an oxymoron to say that two entities are identical. However, in speaking of two natural numbers, '***a*** is identical to ***b***' can mean "***a*** is similar to ***b with respect to*** grouping structure" (abbr. SIM(***a***,***b***,S) although "***a*** is different from ***b with respect to*** location." (abbr. DIFF(***a***,***b***,L). 4 times 3 ***equals*** 12 and also 6 times 2 **equals 12**, but

'4 times 3' means (((()()())(()()())(()()())(()()()))

. '6 times 2' means ((()())(()())(()())(()())(()())(()()));

so 4 times 3 is different than 6 times 2 with respect to grouping structure. All of the following are quite different with respect to grouping structure: 42×11, 3×154, 77× 6, 21×22 , 33×14, 7× 66; though all are equal to each other and to = 462. Indeed, even the commutative law of multiplication

(x×y) = (y×x) hides differences in grouping structure.

2.22 If we wish to think of component entities within a compound 15 x ntity as distinguisble from other entities within that compound (without saying in what respect they are distinguished) we can subscript '()' or 'e' with a numeral Subscripted numerals do not represent numbers (the alphabet letters could do as well)**;** they are merely a familiar set of signs that are different as signs and their differences as signs are used to represent the fact that certain entities are different (without saying *how* they differ) from other entities in a group of entities **.**

============1/13/04===>

In every positive integer a necesssary presupposition is that each unit is distinct from and independent of any other unit in the group. The symbol '(()()())' for 3 or any other positive integer satisfies this assumption**;** every unit is distinct with respect to location in the symbol from every other unit**.** If you cound 4 things, then discover that the first is the same thing as the third you counted, then you say there were really only 3 things. In dealing with number we assume this kind of mistake is not made, or if made is corrected.

In adding numbers, whether positive integers or compound numbers, the assumption must be that each unit listed is distinct from and independent of every other unit within each positive integer and/or in any other positive integer in the compound.

In multiplying, i.e., in replacing each unit in a given possible integer with another positive natural number, it must be assumed that every unit introduced by this process is distinct from every other unit introduced in the product.

The symbols with parentheses for addition and multiplication satisfies this assumption**;** every unit is distinct with respect to location within the symbol from every other unit**.**

Ratios are relations between natural numbers. Ratios differ from natural numbers with respect to the requirement of distinctness of units. Units in each of the natural numbers that are the numerator and the denominator must be distinct from all other members of that natural number. But the two members of the ratio may, or may not, refer to the same, or part of the same, group of individuals. For example the numerator may be the number of individuals in some class, and the denominator may be the number of individuals in a sub-set of that class -- or vice versa.

Different units within a natural number must represent distinct entities. But a relation is not a natural number. It is a relation between natural numbers. A relation is not a unitary thing. (1:2) is a different ratio than (4:8) but the units being talked about in (1:2) may be the same units that are being talked about in (4:8).

12 stands in the relation '(...= the product of ....)' to 6 distinct of positive-integer-multiplication-pairs, namely, to (12 × 1), (1 × 12), (2 × 6), (6 × 2), (3 × 4), (4 × 3). The relation we are talking about here is the natural-number-equality relation between an integer and multiplication-pairs of positive integers. There are just 6 distinct instances of it. Natural-number-equality (what is ordinarily called simply equality) is a relation relation that is always talking aobut exactly the same entities in both terms. In other words, the twelve distinct ratios

(12:(12 × 1)), (12:(1 × 12), (12:(2 × 6)), (12:(6 × 2)), (12:(3 × 4)), (12:(4 × 3)),
((12 × 1):12), ((1 × 12):12), ((2 × 6):12), ((6 × 2):12), ((3 × 4):12), ((4 × 3):12),

all are instances of the relation *natural-number-equality* and as such are talking about the same unit entities in denominator and numerator.

12 stands in the relation (...is greater than...) to 11 positive integers. It stands in the relation (...is less than...) to an infinity of positive integers (We don't say "an infinite number of" since there are no infinite natural numbers). But we can identify as many true individual cases of "12 is less than the natural number,..." as we wish.

Two ratios are the same (natural-number-equal) just in case the numerators are (natural-number-)equal and the denominators are (natural-number-)equal. The ratios (a:b) and (c:d) are proportionately equal, if and only if (a × d) is (natural-number-) equal to (b × c)..Thus, ((2 × 3): 7) = (6:(3+4)) and (6:7) $_p$= ((56-8):(7× 9)). Two ratios are identical if and only the numerators are identical and the denominators are identical. If numerators (denominators) are synoymous they are identical numerators (the denotations are identical) although the definiens and definiendum are not identical (they are never identical in significant synonymies).

===========1/10/04====>

Compare to metric system, based n decimal system: 10 cents = 1 dime; 10 dimes = $1.
Old English: 4 farthings= 1 penny, 12 pennies = 1 shilling 20 shillings = 1 pound
1 pound = 20 shillings = 240 pennies = 960 farthings
1 shilling = 12 pennies = 48 farthings
1 penny = 4 farthings

=================note, **5/30/03** ===>

The parenthetical notation for numbers **displays** numbers, as the **arabic** numerals do not**.** This is *one dimensional* display**.** In *two dimensions*, circles would replace pairs of parentheses, and enclosure structures would be circles inside circles with the smallest circles inside any given circles being unit circles**.** If all unit circles had the same size that would help clarify, but this is not necessarily the case**.** In *three dimensions* all enclosure structures would be spheres (transparent ones?) Again the smallest spheres inside any given sphere would be unit spheres, i.e., unit spheres would be the spheres without anything inside them. In *time* some sphere-structures persist over time unchanged while other sphere-structures disappear or come into being. The concept of space involves the concept of fixed sphere-structures (don't come into being or disappear) at any given "instant" or in any case, no change unless time is brought in.

Finally in 3-dimentional topology the shapes and sizes are not fixed, only the relationship of shapes (bounded 3-D entities) as being inside or outside each other, and as varying in terms of how many (PI) shapes are taken to be immediately inside another, are relevant. All two dimensionsal, three dimensional, and topologically three dimensional parathesis-, circle-, sphere- or (topological) region-structures can be put into one-one correspondence with the parenthetical displaying of positive integers and numbers.

==

What is the advantage of defining numbers in terms of certain kinds of groupings of parentheses? It gives an intuitive meaning to the operations of adding, multiplying and raising to powers. One can inspect the parenthetic expressions and prove from the meanings of 'plus' and, 'times', and 'equals'**....** rather than postulate...

=========[3]

2.23 <u>The theory of well-formed parenthetical structures.</u>(Structures of compound numbers)

2,231 <u>Positive Iintegers</u>

The theory of well-formed parenthetical structures. (I.e., signs composed of pairs of parentheses)

a) Df' 'PI' (i) '(())' is a wfPI ('wfPI' for 'a well-formed Postive Integer')

(ii) if [(â)] is a wfPI & [(ã)] is a wfPI, then [(â ã)] is a wfPI

Thus 1) '(())' is a wfPI [Df 'WfPI', Clause(i)]

2) '(())' is a wfPI [Df 'WfPI', Clause(i)]

3) ('(())' is a wfPI & '(())' is a wfPI) [Adj, 1),2)]

4) '(()())' is a wfPI [Df 'WfPI', Clause(ii1),3)]

5) '(())' is a wfPI & '(()())' is a wfPI [Adj, 2),4)]

6) '(()()())' is a wfPI [Df 'WfPI', Clause(ii),2)]

7) '(())' is a wfPI & '(()()())' is a wfPI [Adj, 2),6)]

8) '(()()()())' is a wfPI [Df 'WfPI', Clause(ii),7)]

9) '(()()))' is a wfPI & '(()()())' is a wfPI [Adj, 4),6)]

10) '(()()()())' is a wfPI [Df 'WfPI, Clause(ii), 9)]

---

[3] (I don't want the following definition of wfps (well-formed parenthetical structure) Because it leads to 7) '(()(()()))' is a wfps, which confuses sum & produce structures.

(i) '()' is a wfps

(ii) if â is a wfps then [(â)] is a wfps

(iii) if [(â)] is a wfps & [(ã)] is a wfps, then [(â ã)] is a wfps

Thus 1) '()' is a wfps [Df 'Wfps', Clause (i)

2) '(())' is a wfps [Df 'Wfps', Clause(ii),1]

3) '(())' is a wfps [Df 'Wfps', Clause(ii),1)]

4) ('(())' is a wfps & '(())' is a wfps) [Adj, 2),3)]

5) '(()())' is a wfps [Df 'Wfps', Clause(iii),4)]

6) '((()()))' is a wfps [Df 'Wfps', Clause(ii),5)]

7) '(()(()()))' is a wfps [Df 'Wfps', Clause(iii), 3) and 6)]

The method above does not allow this kind of PS 7) "'(()(()()))' is a wfps".

b) Naming Positive Integers directly

Df ‘1’: ‘1’ Syn_{df} ‘(())’
Df ‘2’: ‘2’ Syn_{df} ‘(()())’
Df ‘3’: ‘3’ Syn_{df} ‘(()()())’
Df ‘4’: ‘4’ Syn_{df} ‘(()()()())’
Df ‘5’: ‘5’ Syn_{df} ‘(()()()()())’
Df ‘6’: ‘6’ Syn_{df} ‘(()()()()()())’
Df ‘7’: ‘7’ Syn_{df} ‘(()()()()()()())’
Df ‘8’: ‘8’ Syn_{df} ‘(()()()()()()()())’
Df ‘9’: ‘9’ Syn_{df} ‘(()()()()()()()()())’
Df ‘X’: ‘Ten’ Syn_{df} ‘(()()()()()()()()()())’
Df ‘XI’: ‘Eleven’ Syn_{df} ‘(()()()()()()()()()()())’
Df ‘XII’: ‘Twelve’ Syn_{df} ‘(()()()()()()()()()()()())’
Etc,

A single sign, declared to be the direct name of some Pos Int, qua sign, is called a *digit***.**

2,232 Positive Numbers

Df**.** ‘Nn’ (‘Nu’ for “number”)
(i) If â is a wfPI then â is a wfNu
(ii) If â is a wfNu, then [(â)] is a wfNu
(iii) If â is a wfNu & ã is a wfNu, then [(â ã)] is a wfNu

Thus 11) ‘(())’ is a wfNu [Df ‘WfNu’, Clause(i)]
12) ‘(())’ is a wfNu [Df ‘WfNu’, Clause(i)]
13) (‘(())’ is a wfNu & ‘(())’ is a wfNu) [Adj, 11),12)]
14) ‘((()))’ is a wfNu [Df ‘WfNu’, Clause(ii),2)]
15) ‘((())(()))’ is a wfNu [Df ‘WfNu’, Clause(iii),13)]
16) ‘(())’ is a wfNu & ‘(()())’ is a wfNu [Adj, 2),4)]
17) ‘((())(()()))’ is a wfNu [Df ‘WfNu’, Clause(iii),16)]
18) ‘(())’ is a wfNu & ‘(()()())’ is a wfNu [Adj, 2),6)]
19) ‘((())(()()()))’ is a wfNu [Df ‘WfNu’, Clause(ii),18)]
20) ‘(()()))’ is a wfNu & ‘(()()())’ is a wfNu [Adj, 4),6)]
21) ‘((()())(()()()))’ is a wfNu [Df ‘WfNu’, Clause(ii),20)]

I think there is no way to get ((()(x))) or ((x)()()()()) etc**.**, where ‘(x)’ is a PI or Nu

Df**.** [((â) + (ã))] for [((â)(ã))]
Df [((()) x (()))] for [((()))]
Df [((â) x (ã))] for [((â)(ã))] ??

==============

2**.**31**.** Numerals as Parenthetic Symbols

The class of pure parenthetical expressions which we shall use to denote simple numbers, or positive integers, then will be simply the class of 1-level parenthetical expressions**:**

(( )), (( ) ( )), (( ) ( ) ( )), (( ) ( ) ( ) ( )), (( ) ( ) ( ) ( ) ( )), (( ) ( ) ( ) ( ) ( ) ( )), etc.,

From now one, for convenience and perceptual clarity we shall abbreviate every 0-level expression, '( )', by the sign 'o'. [Note that this does not signify zero; it is just an abbreviation of an elementary parenthetical expression]. Thus the sequence above becomes, by abbreviation,
(o), (oo), (ooo), (oooo), (ooooo, (oooooo), etc.
and as might be expected we shall interpret these symbols in such a way that they may be said to have the same meanings or referents as the ordinary numerals '1', '2', '3', '4', '5', '6', . . .etc., respectively. Letting 'syn' abbreviate 'is synonymous with' we may express this as:

'(o)' syn '1'
'(oo)' syn '2'
'(ooo)' syn '3'
etc...

However, it should be noted that the left hand expression in all such cases is ideographic with respect to its denotation on our theory, whereas the right hand expression is not. This much, of course, could also be said of the expressions 'h ab.a(a(ab)); 'S(S(S(0)))' or '0', which are used in other formalizations of arithmetic as synonymous with '3'; though there are differences in the purity of the correlations between sign and denotatum, and in our opinion parenthetical notation is isomorphic to its denotatum in more ways than any of the others.

Using the word 'numeral' for any expression in a syntactical system which is intended to denote a number (as we are using 'number') we may now distinguish simple numerals (for positive integers) from compound numerals, and define the former recursively as follows:

D1. 'x is a simple numeral' syn '(either x is '(( ))' or
x is [(y ( ))] and [y] is a simple numeral)'

From this it follows that all and only 1-level parenthetical expressions will be simple numerals, (standing for positive integers) in the primitive notation of our system. Having defined simple numerals, we may define numerals in general as follows:

D2 'x is a numeral' syn 'Either (i) x is a simple numeral
or (ii) x is a result of replacing each occurrence
of '( )' in some numeral y, by another numeral.'

Clause (ii) may be made the definition of a compound numeral:

D3. 'x is a compound numeral' syn 'x is a result of replacing each occurrence
of '( )' in some numeral y, by another numeral.

From this definition it follows that

T2. If x is a numeral, then [(x)] is a numeral.
Proof: 1) '(o)' is a numeral [by D1]
2) let x be a numeral [Assumption]
3) [(x)] is a result of replacing

an occurrence of ‘o’ in ‘(o)’
by a numeral x [inspection, and 2]
4) [(x)] is a numeral [3), D3]
5) T2 [2) - 4),conditional proof]

T3. If x is a numeral and y is a numeral, then [xy] is a numeral.
Proof: 1) similar, except step 1) is: ‘(oo)’ is a numeral.

These theorems however, do not exhaust the ways in which new numerals are constructed; not only could there be a theorem of the same sort as T2 and T3 for every other simple numeral (or positive integer); each new numeral thus construed will have a finite set of ‘o’s all of which may be replaced. [(by D2, (ii)’ by numerals; and these in turn are subject to the same process, etc.. It will remain the case that simple numerals (denoting positive integers) are all and only the 1-level numerals, and compound numerals (denoting positive integers) are all and only the 1-level numerals, and compound numerals (denoting compound numbers) are all and only numerals of 2nd and higher levels. Thus the compound numerals will include, among others,

I	((o)),(((o))),((((o)))),. . . ((oo)),(((oo))),((((oo)))),. . . ((ooo)),(((ooo))),((((ooo)))),. . .	[By iterated use of T2 on ‘(o)’ and ‘(ooo)’]
II	((o) (o)), ((oo) (o)), ((o) (oo)), ((ooo) (o)),. . . ((o) (oo)), (o) (ooo)), ((oooo) (o)), ((ooo) (oo)),. . . ((oo) (ooo)), ((o) (oooo)), ((ooooo) (o)),. . .	[By application of T3 to pairs of ‘(o)’,‘(oo)’, ‘(ooo)’,(oooo),etc.]
III	(((o) (o))), ((((o) (o)))), (((((o) (o))))),. . . (((o) (o))), ((((oo) (o)))), (((((oo) (o))))),. . . (((o) (oo))), ((((o) (oo)))), (((((o) (oo))))),. . .	[By iterated application T2,to members in group II gotten by T3]
IV	(((o) (o)) (o)), ((o) ((o) (o))), (((oo) (o)) (o)),,. . . . (((o) (oo)) (o)), (((o) (o)) (oo)), (((oo), (o)) (o)),. . . ((o) ((oo) (o))), ((o))), ((o) ((o) (oo))),. . . .	[By T3 on pairs with one from Group II and the other a simple numeral]

In addition to these expressions, we shall be referring to the following ones later:

V	((ooo) (ooo))	[By T3, ‘(oo)’ for ‘y’, ‘(ooo)’ for ‘z’]
	((oo) (oo) (oo))	[By T3,’(ooo)’ for ‘y’, ‘(oo)’ for ‘z’]
	(((ooo) (oooo)) ((ooo) (oooo))) etc.	
	(((ooo) (ooo)) ((oooo) (oooo)))	
	(((oo) (oo)) ((oo) (oo)))	

2.32 Compound Numerals, Sums and Products

Next, we introduce the sign ‘+’ as follows:

D4. Provided [(x)] and [(y)] are numerals, [((x)+(y))] syn [((x) (y))]

This means, syntactically, that if z is a numeral and ‘)) ((’ occurs in z, it may be replaced by ‘))+((’**;** or again, that if one numeral is followed immediately by another, within a third, we may place ‘+’ between the first and the second**.** Thus group II of the numerals above, may be re-written, according to D4, (and remembering that ‘o’ abbreviates ‘( )’) as

((o)+(o)), ((oo)+(o)), ((o)+(oo)), ((ooo)+(o)), **. . .** etc**.**

and then, replacing the simple numerals by their abbreviations, we get for all of group II,

(1+1), (2+1), (1+2), (3+1), **. . .**
(2+2), (1+3), (4+1), (3+2), **. . .**
(2+3), (1+4), (5+1), **. . .**

Applying this rule to components of third-level expressions in group IV we get

(((o)+(o))+(o)), ((o)+(o))), (((oo)+(o))+(o)), etc**.**

So that, with abbreviations for simple numerals we get in Group IV as a whole

((1+1)+1), (1+(1+1)), ((2+1)+1), **. . .**
((1+2)+1), ((1+1)+2), (2+(1+1)), **. . .**
(1+(2+1)), (1+(1+2)), **. . .**

All of which are familiar numerical expressions in ordinary language**.**

Definition D4, however, does not help us to translate expressions in group I into some ordinary expression, since these expressions, gotten by T2, contain no instances of ‘)) ((’ , Since group III contains expressions gotten by T2 alone in the same way, group III expressions are also not completely translatable by this convention**.** What we need then is another definition**.** This one is related semantically to multiplication and to certain

D5**.** [x is (y**.**z)] syn (i) [x,y, and z are numerals and
every n-th level component of x is an occurrence
of the same n-th level numeral z, and
(ii) if every n-th level component of x is replaced by
‘o’ the result is a numeral y**.**]

In an expression [(y**.**z)], y will be said to denote the multiplier and z the multiplicand (number which is multiplied)**.** We shall later show that the laws of commutation and associativity of multiplication are provable in the form ((x**.**y) = (y**.**x)) and (x**.**(y**.**z)) = ((x**.**y**.**)**.**z)’ because of the meaning we shall give to ‘=’ But we do not want to say that (y**.**z**.**)’ syn ‘(z**.**y)’, i**.**e**.**, that ‘(y**.**z)’ denotes the same number as ‘(z**.**y)’**.** Thus “denotes the same number as” does not mean the same as ‘=’ in our system, and consequently ‘[(y**.**z) is the same number as (z**.**y)] ’will not be a law of our arithmetic**.** (This stems from our fundamental departure in the philosophical concept of what a number is)**.** Thus the term ‘multiplier’ will always refer to the first term in the expression ‘(y**.**z)’ and ‘multiplicand’ to the second, and the two terms are not interchangeable**.** Our motivation here, as well as our definition above, may be made clearer by considering the first two expressions in Group V**:**

1**.** In D5 let s be‘((ooo) (ooo))’**:** then S is a 2-level numeral and

(i) every 1-level component of x is an occurrence of
a 1-level numeral z, which is '(ooo)'
(ii) if every 1-level component of x is replaced by 'o', the result
is a numeral y,which is '(oo)'
Hence, by (i), (ii) and D5, x is the same as (y.z) or
'((ooo) (ooo))' syn '(2.3)'

2. In D5 let x be '((oo) (oo) (oo))': then x is a 2-level numeral and
(i) every 1-level component in x is an occurrence of the 1-level
numeral z, which is '(oo)'
(ii) if every 1-level component of x is replaced by 'o', the result I
is a numeral y, which is '(ooo)'
Hence, by (o), (ii), and D5, '((oo) (oo) (oo))' syn '(3.2)'

Remembering the ideographic aspect of our parenthetical notation, it makes sense to say that there is a difference between multiplying a group of three things twice, and multiplying a group of two things three times. This intuitively clear difference is preserved in the syntactically clear distinction between the hierarchical structures of '((ooo) (ooo))' and '((oo) (oo) (oo))', and this difference is further preserved by the convention that '(2.3)' is not synonymous with '(3.2)'; they abbreviate two different syntactical structures, which in turn, on our theory, ideographically denote different "numbers". In general, the left hand component in '(y.z)', the multiplier, denotes a higher level over-all structure of x, while the right-hand component, the multiplicand, denotes the structure of all the lower level components of the over-all structure. Commutatively and associativity of multiplication will come about through our definition of '=', later.

It is important to remember that in the theory of positive integers D5 does not allow every number to be a product of two others. None of the simple numbers, or positive integers, will be the same as a product of two numbers; for numbers have the grouping properties of numerals, and simple numerals have just two levels, o-level and 1-level. It is not the case that every o-level component, 'o', is an occurrence of some numeral z (since 'o' is not a numeral at all), and though every simple numeral is 1-level and thus a numeral, if, by D2, (ii) every 1-level component of a simple numeral is replaced by 'o', the result is just 'o' and this again is not a numeral x. There will be many other cases of numerals which do not denote products or multiplications in the theory of positive integers: '((oo) (ooo))' will not, because it is not the case that every 1-level component of '((oo) (ooo))' is and occurrence of the same 1-level numeral, z, and the o-level and 2-level components are subject to the same objections mentioned with respect to the o and 1-level components of simple numerals above. Other numerals may have components which are products, though not products themselves; e.q., '((oo) ((ooo) (ooo) (ooo) (ooo)))' is not, as a whole a product, but can be abbreviated as '(2+(4.3))' with its right-hand main component a product. Note that numbers which are not products are not necessarily prime numbers. A prime number will be defined later as a number which is not equal to any product except itself times one. Thus (2+(4.3)) is not a prime number though (2+(4.3)) is not itself a product; it will be shown later to be equal to (2.7) which is a product, hence it is not a prime number. On the other hand, though we shall show that ((oo) (ooo))=(ooooo), i.e., (2+3) = 5, it will also be the case that (2+3), and 5, can not be shown to be equal to any products other than (1.5) and (5.1), and thus may be called prime numbers.

This brings us to the expressions in Groups I and III, which can be translated into familiar arithmetic language by means of D5, as follows:

3. In D5 let x be '((o))': then x is a 2-level numeral, and
(i) every 1-level component of x is an occurrence of the same 1-level numeral z, which is '(o)'.
(ii) the result of replacing every 1-level component of x in x by 'o' is a numeral y, which is '(o)'.
Hence, by (i) (ii) and D5,'((o))' syn '((o).(o))'
Hence, by definition of '1','((o))' syn '(1.1)'

By similar reasoning, applied not only to the whole, but to components in turn we get the following translations for the expressions in Group I:

(1.1), (1.(1.1)), (1.(1.(1.1))), . . .
(1.2), (1.(1.2)), (1.(1.(1.2))), . . .
(1.3), (1.(1.3)), (1.(1.(1.3))), . . .

By similar analysis, the expressions in group III are brought into familiar language.

'(((o) (o)))'syn '(1.(2.1))' ; ' ((((o) (o))))' syn' (1.(1.(2.1)))' etc..

and the group as a whole becomes translatable as ...

(1.(2.1)), (1.(1.(2.1))), (1.(1.(1.(2.1))), . . .
(1.(2+1)), (1.(1.(2+1))), (1.(1.(1.(2+1)))), . . .
(1.(1+2)), (1.(1.(1+2))), (1.(1.(1.(1+2)))), . . .

Returning now to the third and fourth expressions in Group V,

'(((ooo) (oooo)) ((ooo) (oooo)))' syn '(2. (3+4))' syn '((3+4)+(3+4))'
'(((ooo) (ooo)) ((oooo) (oooo)))' syn '((2.3)+(2.4))' syn '((3+3)+(4+4))'

The <u>difference</u> between these two expressions is clear. It remains to be shown how we define '=' so that we can prove (2.(3+4))=((2.3)+(2.4)), or more generally, the distributive law of multiplication over addition, namely (x.(y+z))=((x.y)+(x.z)), of which (2.(3+4))=((2.3)+(2.4)) in an instance. The fifth expression in Group V is variously translatable as follows:

'(((oo) (oo)) ((oo) (oo)))' syn '((2+2) + (2+2))' syn '(2. (2+2)syn'((2+2).2'
syn '((2.2) + (2.2))' syn '(2. (2.2))'syn '2^{3}'

There is a certain naturalness, I believe, in the foregoing accounts of '.' and '+' for multiplication and addition. When we think of multiplying one number of things (the multiplicand) by another (the multiplier), it seems much like thinking of the multiplier as an organized grouping in which we replace the each zero-level component by the multiplicand, Thus,

1) If I am considering a package which has 4 apples
in ((a) (a) (a) (a)) each package [multiplicand]
2) and I decide to buy 3 of these packages (( ) ( ) ( )) [multiplier]
3) then what I decide to buy can be represented by replacing each 0-level component of the multiplier by the multiplicand:
(((a) (a) (a) (a)) ((a) (a) (a) (a)) ((a) (a) (a) (a)))

But our definition handles not only simple cases like that above in which both multiplier and multiplicand are simple numbers (positive integers) but also cases where both are compound numbers, e.g., (1+2) and (3+4). Thus suppose I want to buy seven metal signs, m, each sign having two screws, s, accompanying it, i.e., I want to buy items of the form '((m) ((s) (s)))'-the multiplicand. But it turns out that one store has only three - (ooo) - of these items and another has just four - (oooo), then what I want is (((ooo)+(oooo)) . ((m)+(s))) or ((3+4) . (1+2)), or (3 . (1+2))+4 (4.1(1+2)). This, however, by our rules, is simply an abbreviation of the following (using '(m(ss))' for '((m)+((s)+(s)))'):

Multiplier.multiplicand: (((ooo) (oooo)) . (m(ss)))
or
((((m(ss)) (m(ss)) (m(ss))) ((m(ss)) (m(ss)) (m(ss)) (m(ss))))
(a store 1) (at store 2)

In general, I believe the concepts of multiplication and addition suggested ideographically by our parenthetical notation for numerals is more natural and closer to ordinary intuitive understandings of elementary numbers and arithmetic than any other alternative with which I am familiar.

A full account of the notation for exponentiation, 'x^y' must await a fuller discussion of '='. Although it might appear that we could define 'x^{4}' as '(x.(x.(x.x)))', and 'x^{5}' as '(x.(x.(x.(x.x)))), etc., it is not clear how the expression '4' and '5' would be used in such cases. According to our theory '4' syn '(oooo)' and '5' syn '(ooooo)'. In the definition of multiplication there is a clear sense in which the structure '(oooo)' is embedded in, or analytically part of, the structures of the expressions abbreviated by '(4.5)' and '(5.4)' (though the role played by '4' in each of these cases is different). But in an expression like '3^{4}', or 3^{5}', there is no place where the numerals '4' or '5' are either components, or embedded in the structures, of the numeral denoted. Thus we shall defer consideration of how to translate expressions into the notation of exponentiation until later.

2.4 The Adequacy of Parenthetical Notation.

The definitions given above for '1', '2', '3', . . ., '+' and '.' are adequate to permit the translation of every numeral in parenthetical notation into a familiar arithmetical expression. This is clear from the definition, D2 of 'x is a numeral'. Clause (i) yields all and only 1-level expressions; but all of these are translatable into the familiar notation of positive integers, i.e., '1', '2', '3', . . .etc..Clause (ii) yields a new numeral x from an old numeral y, by replacing each occurrence of 'o' in y by some or other old numeral. But 'o' can occur in a numeral only in four ways; '..o<u>o</u>o. . .', '...(<u>o</u>o. . .', '...o<u>o</u>). . .', '. . .(<u>o</u>). . .'. Replacing all occurrences of 'o' in any of these contexts by numerals '(<u>x</u>)' or '(y)' or '(z)' will yield '. . .(y) (<u>x</u>) (z). . .' or '. . .((<u>x</u>) (z). . .', or '. . .(y) (<u>x</u>)). . .' or . . .(x). . .' but since 'x', 'y' and 'z' will be at least one pair of parentheses, yielding sequences of ')) ((' in the first three cases, we get '. . .(y)+(<u>x</u>)+(z). . .', and '. . .((<u>x</u>)+(z). . .' and '. . .(y)+(x)). . .' in these cases by D4, and '. . .(1.(<u>x</u>)). . .' in the fourth case by D5. Thus by induction it will follow that all numerical expressions in parenthetical notation are translatable into familiar notation.

It can also be proved easily enough, that all familiar arithmetic expressions which ‘are’ constructible according to the ordinary rules out of numerals for the positive integers, ‘+ ‘.’ , and parentheses, can be translated back into a unique, purely parenthetical notation of our system. Sometimes, to be sure, there are several familiar expressions which will all translate back into the same parenthetical notation- the same element in the domain of grouping theory. Thus, since (2.2) is identical to (2+2), each of ‘((2+2)+(2+2))’, ‘((2.2)+(2.2),and ‘2^{3}' are translatable into the single, unique parenthetical expression ‘(((( ) ( )) (( ) ( ))) ((( ) ( )) (( ) ( )))). In these cases we can say not only that the familiar expressions denote equal numbers, but that they all denote the same number. I.e., not only will it turn out that ((2+2)+(2+2))) = (2.(2.2)) but ‘also ‘((2+2)+(2+2))’ syn ‘(2.(2.2))’; remember that while (2.3)=(3.2) it is not the case that ‘(2.3)’ syn (3.2)’; i.e., the two don’t denote the same object to our domain. Once again, this underscores the new interpretation of ‘=’ in arithmetic as well as a new concept of number. Another interesting example of the same point is

‘((((oooo) (ooooo)) ((oooo) (ooooo))) (((oooo) (ooooo)) ((oooo) (ooooo)) ((oooo) (ooooo))))’

which is translatable into familiar notation by either ‘((2+3) . (4+5) or ‘((2.(4+5))+(3.(4+5))’. Thus the two latter terms not only denote equal objects, but they are synonymous, denoting the same object. By contrast the following instance of the distributive law, ‘(2.(4+5)=((2.4)+(2.5))’ represents an equality, though ‘(2.(4+5)) and ‘((2.4)+(2.5))’ do not denote the same object in our domain.

Broadly speaking, then, the relation between the set of familiar expressions built up from parentheses, ‘+’, ‘.’ and the numerals for positive integers, and the set of numerals in our parenthetical notation, is a many-one relation, a function from the familiar expressions onto the parenthetical notation for numerals.

In the next section, after introducing ‘=’ we shall show that parenthetical notation is also adequate for complete translation into and out of the symbolism which has been used in the formalizations of a theory of positive integers by mathematical logicans using symbols for the successor function, the addition function and the multiplication function. And indeed we shall examine closely the connection between parenthetical notation and its results and the sub-class of elementary number-theoretic functions known as general recursive functions. But first we must define ‘=’ and other relationships between objects in the domain of elementary number theory.

Our purpose up to this point has been to define a domain of objects, and a set of terms capable of denoting uniquely each number of this domain. These objects, we propose, are the proper objects of the elementary arithmetic of positive integers. We shall further propose that only such statements about these objects or about relationships between members of this domain as are decidable by reference to and analysis of the nature of the objects as defined should be counted as mathematically true or mathematically false statements. And we shall show that there are a variety of formally definable relations between members of this domain, some of which are recursive relations (having a recursive characteristic function) which are not, on this account, strictly mathematical or arithmetic relationships. At this point, of course, we go to the heart of the issue; whether recursive relations are necessarily mathematical or arithmetical relations. If they are not, we shall, then Church has not shown that there are arithmetic truths which are undecidable even granting his thesis.

2.5 Mathematical Properties and Relations is the domain of Elementary Arithmetic.

Set theoretical approaches to mathematics proceed from an interpretation of ‘=’ which has been expressed as follows: “In general, the equality sign is placed between two expressions to indicate that these expressions are names or descriptions of one and the same object” (Hankin et al, Retracing Elementary Mathematics, Macmillan, 1962 p 5). On such a view ‘(2+1)=3' must be interpreted as saying that all (tokens of the type ‘(2+1)' and all tokens of the type ‘3' refer to name or describe just one and only one object in the universe. What is this strange object? And where can it be found? It is neither here nor there, we are told; or it is everywhere. It fills the universe, being embedded in the collection of all physical objects; or is a denizen of Plato’s strange heaven. It is the class of all classes with just three members; yet none of us have ever seen or apprehended this class - though we may have seen some of its members. Why this immense burden on the imagination? Is this really necessary to have a viable theory of mathematics? Mathematicians are no less finite and human than the rest of us. Can we not get a theory of mathematics which does not impose such flights? Indeed we can. And to show that this can be done is one purpose of this paper. The sign ‘=’ is interpreted in a very ordinary, yet precise way, based on the fact that human beings, in single moments, can determine particular similarities. ‘(2+1)’ neither means, nor denotes, the same object as ‘3', ever. To say that ‘(2+1)=3' is always and logically true; but true by virtue of a different meaning of ‘=’ than that just mentioned.

In elementary arithmetic the primary relations we are interested in are those of (i) arithmetic equality, signified by ‘=’, and (ii) arithmetic inequality, signified by ‘(’. Both are binary relations. If a system of arithmetic is to be complete, then given any two objects x and y, in the domain, either [x=y] will be true or [x (y] will be true (an application of the law of excluded middle). If it is to be consistent, then no two objects in the domain, x and y, can be such that both [x = y] and [x (y] is true. The primary job of arithmetic is computation - to find numbers which are equal to certain other compound or simple numbers. Addition and multiplication tables summarize such equations in tabular form, and algorithms for multiplying and adding large numbers or compound numbers have as their terminal goal numbers which are equal to the products or sums of the initial numbers. The five fundamental laws of arithmetic are all algebraic equations:

(x+y)=(y+x)	[Commutation of x]
(x+(y+z))=((x+y)+z)	[Association of +]
(x.y)=(y.x)	[Commutation of .]
(x.(y.z))=((x.y).z)	[Association of .]
(x.(y+z))=((x.y)+(x.z))	[Distribution of . over +]

Further, in elementary arithmetic, given equality and the concepts of functions ‘+’ and ‘.’, we can define’‘<’, ’>’, ‘< ‘, ‘< ’, ‘. .is the successor of. .’, ‘. . is a power of . .; ‘. . is a factor of . . ; . . is divisible by . .; and other relations, as well as such functions as ‘S’ for the successor functions, ‘-’ for the subtraction function and ‘ m’ for the division function, etc., in currently familiar ways. Thus we will deal next with the concept of arithmetic equality in our theory, leaving until later a comparison of this account with the more usual set-theoretic treatments stemming from Frege and Russell.

2.51 Equality versus Identity

Two different expressions will be said to denote the same number if and only if they are synonymous with the same numeral in our parenthetical notation. We shall use the expression ‘xIy’ to express the notion in ‘x is identical with y’. Or ‘x is exactly the same as y’; in this case, of course, we are concerned with expressions which assert that some x is the same number, in our sense of number, as some y. We shall use ‘I’ as a primitive relation and introduce ‘=’ for arithmetic equality, by definition. Actually, ‘I’ is closer to the way ‘=’ has usually been used in set theory or predicate logic with identity. But since our purpose is to draw a distinction between identity (I) and arithmetic equality, (=) and since the sign ‘=’ has been used much longer and more widely for arithmetic equality than it has for logical or set-theoretical identity, we shall reserve ‘=’ for the defined term, rather for identity itself. Thus we shall suppose that ‘I’ has been introduced as a prior logical primitive. A second relation will be introduced: one number will be said to be reducible to a second number, if the result of eliminating all intermediate groupings from the first is identical with the the result of eliminating all intermediate groupings from the second. An intermediate pair of parentheses is one which is neither the outer most pair nor an inner most pair. We will use the symbol ‘xRy’ for ‘x is reducible to y’ in this sense. Finally, we will say that two numbers are equal to each other if there is some number both are reducible to, i.e., assuming x,y, and z range over numbers.

D6. [x=y] syn [(Ez) (xRz,yRz)]

This definition of arithmetic equality conforms neatly to ways in which elementary arithmetic is used. If I buy three packages of four apples each, and someone asks how many apples I bought, I literally disregard the intermediate groupings (the groupings in packages) and consider only the elementary objects (the apples) and the over-all grouping, and answer ‘12'. In short I initially have something like ((aaaa) (aaaa) (aaaa)), three packages of four apples, and to answer the question, I disregard the intermediate groupings to get (aaaaaaaaaaaa), and answer twelve. Thus when I calculate (3.4)=12, or (4+4+4)=12, I am literally “reducing” ‘((oooo) (oooo) (oooo))’ to ‘(oooooooooooo)’ by dropping out all the intermediate groupings represented by pairs of parentheses which are neither zero-level nor the top n-th level parentheses in the expression.

It is immediately clear that if we eliminate all intermediate parentheses from any numeral, what will be left will be a simple numeral representing a positive integer. Further, for any given compound numeral, there will be one and only one simple numeral to which it can be reduced; for the number of zero-level expressions will not be altered in any way by the numbers of intermediate groupings involved. Thus the reducibility relation is a many-one relation, or function; i.e., for any given numeral, simple or compound, there will be one and only one numeral to which it is reducible, though there will be infinitely many non-identical compound numerals (numbers) which will be reducible to any given simple numeral (positive integer). This instead of saying that compound expressions like ‘(2.(3+4)) is identical with ‘(6.(1+3))’, denote or stand for the same positive integer, 24, or (oooooooooooooooooooooooo), and thus that (2.(3+4)) is identical with (6.(1+3)), we say that ‘(2.3+4))’ and ‘(6.(1+3))’ are both reducible to 24, although 24, (2.(3+4)) and (6.(3+1)) are all different numbers, i.e., are non-identical.

The relation of identity will hold between any numeral in parenthetical notation and itself, of course. Thus we may assert

(ooo)I(ooo)	3I3
((oo) ((oo) (oo)))I((oo) ((oo) (oo)))	(2+(2.2))I(2+(2.2))
(((oo) (oo))I(((oo) (oo)) (oo)) (oo))	((2.2)+2)I((2.2).2)
(oooooo)I(oooooo)	6I6

and by virtue of our translations and abbreviations, we may further assert such identities as

(ooo)I3	2^3I((2+2))+(2+2))
((oo) ((oo) (oo)))I(2+(2.2))	2^3I((2.2)+(2.2))
(2+(2.2))I(2+(2+2))	(2.(2.2))I(2.(2+2))

for in all of these cases, the different expressions on either side of an 'I' stand for or denote the same number. But we can not, in our theory, say that (2+(2.2))I6, or even that (2+(2.2)+((2.2)+2), or (2.3)I(3.2), for in these cases, the two terms do not denote the same structured groupings, even though they are equal. Thus identity in our theory preserves groupings and orderings; ordinal numbers and ordinal identities or differences can be expressed by means of our initial definition of the domain, and the relation of identity, I. But elementary arithmetic is cardinal arithmetic, in which linear orders are disregarded by rules of commutation and hierarchical orders are disregarded by rules of commutation and hierarchical orders are disregarded by rules of association.

2.52-Proofs of the Five Fundamental Laws

Since the elimination of all intermediate groupings eliminates any distinctions on which either linear or heirarchical order could be based, the device of treating arithmetic equality in cardinal arithmetic as the identity of the reductions of two numbers, yields strict proofs of the five fundamental laws of cardinal arithmetic. Eventually we must give a strict definition of proof in our system; not having done that, we simply give below an example of a simple proof of an instance of each law.

1. $(n_1+n_2) = (n_2+n_1)$ [Commutation of '+']

Example

. a) let n1 be(2+1), let n_2 be(2.3)	Assumption
b) (((oo) (o)) ((ooo) (ooo))R(ooooooooo)	Def R
c) ((2+1)+(2.3)) R (ooooooooo)	b),Df '+', Df '1'.Df '2',Df '3'.
d) (((ooo) (ooo)) ((oo) (o))) R (ooooooooo)	Def R
e) ((2.3)+(2+1) R (ooooooooo)	d) Df '+' .Df '1' .Df '2'.Df '3'.
f) (Ez) (((2+1)+(2.3)+(2.3))Rz . ((2.3)+(2+1))Rz)	c) ,e), Simp, EG
g) ((2+1)+(2.3))=((2.3)+(2+1))	f) ,def '='
h) $(n_1+n_2)=(n_2+n_1)$	a),g), sub.

2. $(n_1 . n_2) = (n_2 . n_1)$ [Commutation of '.']

Example:

a) let n_1 be 2 and n_2 be 3 — Assumption

b) ((ooo) (ooo))R(oooooo) Def R
c) ((oo) (oo) (oo))R(oooooo) Def R
d) (2.3)R(oooooo) b),Df '.' .Df '2', Df '3'.
e) (3.2)R(oooooo) c),,Df '.', Df '2', Df '3'.
f) (Ez) ((3.2)Rz . (2.3)Rz d),e), Simp., E.G.
g) (2.3)=(3.2) f), Def '='.
h) (n1.n2)=(n2.n1) g), a), sub.

3. <u>$(n_1+(n_2+n_3)) = ((n_1+n_2)+n_3)$</u> [Association of '+']
Example:
a) let n_1 be '2', n_2, be '3', n_3, be '1' Assumption
b) ((oo) ((ooo) (o)))R(oooooo) Def R
c) (((oo) (ooo)) (o))R(oooooo) Def R
d) (2+(3+1)R(oooooo) b), Df '+', Df '1', Df'2', Df' '3'.
e) ((2+3)+1)R(oooooo) c), Df '+', Df '1', Df '2',Df' '3'.
f) (Ez) ((2+(3+1))Rz . ((2+3)+1)Rz) d),e), Simp. &E.G.
g) (2+(3+1)=((2+3)+1) f), df '='
h) $(n_1+(n_2+n_3')) = ((n_1+n_2)+n_3)$ g),a), sub

4. <u>$(n_1 . (n_2 . n_3)) = ((n_1 . n_2 .n_3)$</u> [Association of '.']
<u>Example</u>:
a) Let <u>n_1</u> be '2', n_2 be '3', n_3 be '1' Assumption
b) ((((o) (o) (o)) ((o) (o) (o)))R(oooooo) Def R
c) ((((o) (o) (o)) ((o) (o) (o)))R(oooooo) Def R
d) (2.(3.1))R(oooooo) b), Df '.', Df '2', Df '3', Df '1'
e) ((2.3).1)R(oooooo) c), Df '.', Df '2', Df '3', Df '1'.
f) (Ez) ((2.(3.1))Rx . ((2.3).1)Rz) d), e), Simp., E.G...
g) (2.(3.1))=((2.3).1 f), Df '='
h) $(n_1 . (n_2.n_3))=((n_1.n_2).n_3))$ g), a), sub.

(Actually, it can be proven that (n1.(n2.n3))I(n1.n2).n3); but since it can also be proven that (x) (y) (xIy ; x=y), 4 will follow from that fact as well as the sort of proof suggested above. None of the others of these five laws is also an identity.).

5. <u>$(n_1.(n_2+n_3)) = (n_1.n_2)+(n_1.n_3))$</u> (Distribution of '.' over '+']
<u>Example</u>:
a) Let n_1 be '2', n_2 be '3' and n_3 be '1' Assumption
b) (((ooo) (o)) ((ooo) (o))R(oooooooo) Def R
c) (((ooo) (ooo)) ((o) (o))R(oooooooo) Def R
d) (2.(3+1))R(oooooooo) b), Df '.', Df '+', Dfs '2', '3', '1'.
e) ((2.3)+(2.1))R(oooooooo) c), Df '.', Df '+', Dfs "2', '3', '1'.
f) (Ez) ((2.(3+1))Rz . ((2.3)+(2.1))Rz) d), e), Simp., E.G.
g) (2.(3+1))=((2.3)+(2.1)) f), Df '='
h) $(n_1.(n_2+n_3))=((n_1.n_2)+(n_1.n_3))$ g), a), sub.

Although these do not, of course, give general proofs of these laws, they indicate pretty clearly how such proofs can be gotten. We have used only the simplest, smallest positive integers. But since every substituent for 'n_1 ' will be reducible to a positive integer in steps b) and c) of these proofs, the general proof will follow if we can show that the laws hold for every set of positive integers n_1, n_2 and n_3. But surely this is possible, especially if we develop an appropriate principle of induction.

A principle of induction becomes possible with the definition of the successor relation:

D7. [xSy] syn [x = (y+1)]

Given any numeral n_1, [(n_1+(o))] will be its successor. On this definition, clearly the successor of, e.g., 4, or (oooo), will not be identical with 5, or '(ooooo)', but instead will be identical with (4+1) or ((oooo) (o)); nevertheless, the successor of 4, i.e., ((oooo) (o)), will be <u>equal to</u> 5. I.e., letting 'S '4' mean 'the successor of 4', it is not the case that S'4 I 5, but it is the case that S '4=5. We could, of course, have defined the successor relation so as to make it hold only of positive integers [by [xSy] syn [x is PI and (Ez)(yI'(z)' and xI=(zo)'] 'or by using the second clause of D1; [xS(y)] syn [(y)] is a PI . x I [[(y ( )))]]]. But this would eliminate successors of any compound numbers, and the latter concept plays an important role in the axiomatization of arithmetic. We could have two kinds of successor, one for positive integers and one for compound numbers; but this seems unnecessarily complex. The definition above will serve, I think, all purposes needed, and due to our definition of '=', we can get the law that (x) (y) ((xePI.ySx); yIS'x), even though we don't get (x) (y) ((xePI . ySx) ; yIS'x).

Returning now to Peano's postulates for elementary arithmetic, all of his first eight axioms will be derivable from our definitions of numbers (the ideographic denotata of numerals as defined in parenthetical notation above), of '1', of '+', and of '=', together with principles drawn from predicate logic with identity. His axioms 2,3,4, and 5 simply apply to numbers the following principles of identity in logic:

2. (x) (xIx)
3. (x) (y) (xIy o yIx)
4. (x) (y) (z) ((xIy.yIz) ; xIz)
5. (x) (y) (z) ((xIy . y is an N) ; x is an N)

His first postulate,

1. 1 is an N

translated into our system, says simply that (o) is an number, and since '(o)' is a numeral, and the class of numerals, on our account, stand in one-to-one correspondence with the class of (elementary) numbers, our account satisfies

P1. The postulate P6, says that if any entity is a number then its successor is a number:

P6. (x) (x is an N ; (x+1) is an N)

<u>Proof</u>: 1) x is an N	Assumption
2) (oo) is an N	Def. 1

3) (x(o)) is an N — 1, 2, n of 2 clause (ii)
4) (x+1) is an N — Df '1' Df '+'
5) If x is an N then (x+1) is an N — (2-4) C.P.

and this certainly will follow also from our definition of numerals and T2. Postulate P7 says that successors of equal numbers are equal.

P7. (x) (y) ((x is an N and y is an N); (x=y o ((x+1)=(y+1)))

and this also is subject to a quick and easy proof. For if x and y are equal, then they reduce to the same positive integer (by the definition of 'R' and '='; the successor of each of these will be denoted by a parenthetical expression which adds only one more zero-level expression, so that when the intermediate parentheses are eliminated both will again reduce to the same positive integer, i.e., the integer which is equal to the successor of the integer they both reduced to in the antecedent. Actually, this postulate will need a principle of induction for its derivation. Postulate P8 says simply that 1 is not equal to the successor of any (elementary) number:

P8. (x) (x is an N ; ((x+1) (1))

And this, also, is perfectly obvious in terms of our definitions of 'N', '1' and '='; (o) is not equal to any [(n(o))] where n is a number. The final postulate, is in effect the principle of mathematical induction:

P8. (x) (x is G ; x is an N) . 1 is G . (x)((x is an N . x is G); (x+1) is G));
6 (x) (x is an N ; x is G).

Although this principle will be a meta-theorem of our system, it is not adequate as a postulate of the system due to the fact that we have not defined numbers in our system solely in terms of the successor function. Definitions D1 and D2 do not yield a linear (or strict simple) ordering as the successor relation does in Peano's definition of N based on P6. For clause (ii) of D2 allows an indefinite number of results of performing the operation of "replacing each occurrence of '0' in a given numeral by some or other numeral". Thus this mode of generating the numbers gives at best a partial ordering. Nevertheless, it is possible to find an alternative to P9 which is stronger, and from which P9 may be deduced.[3] Thus we may conclude that given our definition of the relation, as an

[3] 1 Cf. For example, Kleene, Stephen Cole Introduction to Metamathematics, 1952 S 50, in which he indicates how to formulate an appropriate principle of induction for the system of Hans Hermes' "Semiotik Eine Theorie der Zeichengestalten als Grundlage für Untersuchungen von formalisierten Sprachen", Forschungen zur Logik und zur Grundlegung der exakten Wissenschaften, n.s., No. 5, Leipzig, 1938. Hermes definition of 'entity', like our definition of number, involves parenthetic enclosure of any finite series of entities previously established, and thus yields only partial ordering.

operation on compound groupings and the consequent definition using logical identity, I, and predicate logic of '=', our theory of elementary arithmetic will conform to the requirements of

Peano arithmetic, and thus constitute a viable theory of the arithmetic of positive integers.

We could go further and define other functions - the subtraction function, the division function, - and relations like greater than, less than, etc., in familiar ways with the concepts at hand. More ambitiously, we could present a formal axiomatized system, and set about proving this system complete with respect to Peano Arithmetic. However, these tasks are not relevant to our present purpose. Our purpose thus far has been merely 1) to define a domain of objects, called elementary numbers (which may be called a sub-class of the class of organized groupings), 2) to give an inductive definition of a specific set of linguistic expressions to be built up from parentheses '(' and ')' and called 'numerals', 3) to propose that all and only those organized groupings which correspond ideographically one-to-one to these numerals constitute the exact domain of objects that elementary arithmetic is about, and 4) to propose that the relations: and function based on the elimination of all intermediate groups, Rxy, (as distinct from the successor function) together with logical identity and predicate logic, may be adequate for all the relations, beginning with arithmetic equality, '=' needed in elementary arithmetic. This we will examine in the next section. Our final step is to show, if that this much as been granted, then it is possible to define in a very clear way the difference between mathematical relation and certain kinds of relations between numbers or sets of numbers which are contingent and non-mathematical.

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For Digresssion on geometry and mathematics after my account of roots and trigonometric functions

In Euclidean plane geometry we can specify by rules of construction, that if we specify that certain given straight line segments.are to be taken as having a unit length of 1, we can provide a method for constructing in theory precise point on a straight line that is the square root of any positive integer. We can do this by constructing right tirangles and using the Pythagorean theorem..

Construct a rt triangle with sides a and b, and hypotenuse h.

To Construct	Side a,		Side b		Hypoteneuse
@ 2	1	+	1	=	(@ 2)
@ 3	1	+	(@ 3)	=	2
@ 4	@ 2	+	@ 2	=	(@ 4)
@ 5	1	+	(@ 5)	=	2
@ 6	@ 3	+	@ 3	=	(@ 6)
@ 7	@ 4	+	@ 3	=	@ 7
@ 8	@ 4	+	@ 4	=	@ 8
@ 9	@ 6	+	@ 3	=	@ 9
@10	@ 5	+	@ 5	=	@10

Trigonometric functions, also historically came from PythagoreannTheorem on Euclid's Geometry, x2. But while geometrical squares are on the sides of the rigth triangle are proven to be equal in area precisely. The most of the mathematical relations can never be reduced to a rational numbers. The idea that there are "real numbers" for each of these is a fantasy. There are no such numbers - only proximation functions. And they are completely separate from plain geometry. The trigonometric functions are also applicable to the relations between diameters and circles in spherical, or eelliptic geometry, where the Pythagorean Theorem does not apply (except as a limit)

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